MMSE OFDM AND PREFIXED SINGLE CARRIER SYSTEMS: BER ANALYSIS

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ABSTRACT

The single carrier system with cyclic prefix (SC-CP) has been demonstrated to outperform the OFDM system through simulations. In this paper, we consider minimum mean squared error (MMSE) receivers for the SC-CP system. We show analytically that, for uncoded QPSK symbols, the SC-CP system with an MMSE receiver has a smaller bit error rate (BER) than the OFDM system for all SNR.

1. INTRODUCTION

There has been considerable interest in the OFDM (orthogonal frequency division multiplexing) transceiver [1][2] and SC-CP (single carrier system with cyclic prefix) transceiver [3][4][5]. In the OFDM system (Fig. 1), the transmitter is a channel-independent IDFT matrix followed by cyclic prefix insertion and the receiver is a DFT matrix followed by M multiplications, which are the only channel-dependent part of the system. The transmitter of the SC-CP (Fig. 2) system is simply a serial-to-parallel conversion followed by cyclic prefix insertion. The receiver consists of a DFT matrix, an IDFT matrix and M multipliers called frequency domain equalizers.

The SC-CP system share many common features with the OFDM system. The transmitters of both systems are channel independent. This property is very attractive for wireless applications, where the transmitter usually does not have knowledge of the channel and also for broadcasting applications, where there are many receivers, each with a different transmission path. In these applications, the transmitter does not employ bit and power allocation. The SC-CP system has overall complexity the same as the OFDM system. Similar to the OFDM system, ISI in the SC-CP system can be canceled completely using redundant cyclic prefix. In addition, the SC-CP system has several advantage over the OFDM system. The SC-CP system has a much lower peak to average power ratio. Moreover it has been demonstrated *through simulation* that, without channel coding the SC-CP system outperforms the OFDM system [6]. In [7], it is shown *analytically* that the SC-CP system with a zero-forcing receiver is better than the OFDM system for low BER and worse for high BER.

In this paper, we perform analytic analysis of the BER performances of the OFDM and the SC-CP with an MMSE receiver. For QPSK symbols without channel coding, we show analytically that, with MMSE receivers the SC-CP system is always better than the OFDM system. The use of MMSE receivers improves the SC-CP system but not the OFDM system. The MMSE SC-CP system is also more robust to spectral nulls than the OFDM system. Furthermore, the MMSE SC-CP system has the same overall complexity as the zero-forcing SC-CP and OFDM systems.

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2. REVIEW OF THE OFDM SYSTEM

The block diagram of the OFDM system is as shown in Fig. 1. At the transmitter, the input modulation symbols s_k are passed through an $M \times M$ IDFT matrix and a cyclic prefix of length L is inserted for each block of size M. The receiver performs M-point DFT and the outputs of the DFT matrix are multiplied by $1/P_k$ as shown in Fig. 1, where $P_0, P_1, \cdots, P_{M-1}$ are the M-point DFT of the channel impulse response $p(n)$. We assume that the channel is FIR with order $\le$ the prefix length L. In this case the transceiver is ISI free. The receiver outputs x_k (Fig. 1) are equal to the transmitter input s_k in the absence of channel noise.

Assume that the channel noise $\nu(n)$ is complex AWGN with variance $\mathcal{N}_0$ and s_k are QPSK symbols, $s_k = \pm\sqrt{\mathcal{E}_s/2} \pm j\sqrt{\mathcal{E}_s/2}$. The subchannel noises are $e_x = x_k - s_k$. They are uncorrelated and $\sigma^2_{e_k} = \frac{\mathcal{N}_0}{|P_k|^2}$. The average mean squared error $\mathcal{E}_{rr} = \frac{1}{M}\sum_{k=0}^{M-1} \sigma^2_{e_k}$ is

$$\mathcal{E}_{rr} = \frac{1}{M}\sum_{i=0}^{M-1} \mathcal{N}_0/|P_i|^2. \qquad (1)$$

Define the SNR quantity $\gamma = \mathcal{E}_s/\mathcal{N}_0$. The noise-to-signal ratio (NSR) of the k-th subchannel is

$$\beta(k) = \frac{1}{\gamma|P_k|^2}. \qquad (2)$$

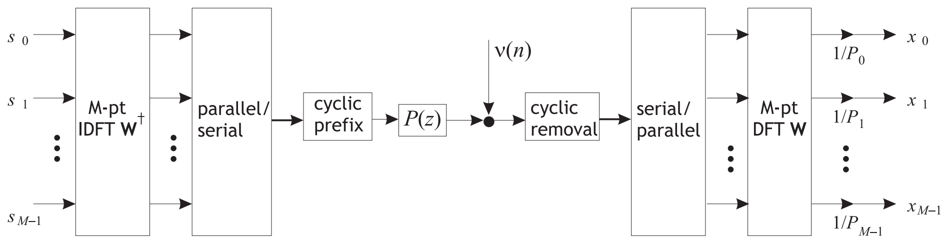

Figure 1: The block diagram of the OFDM system.

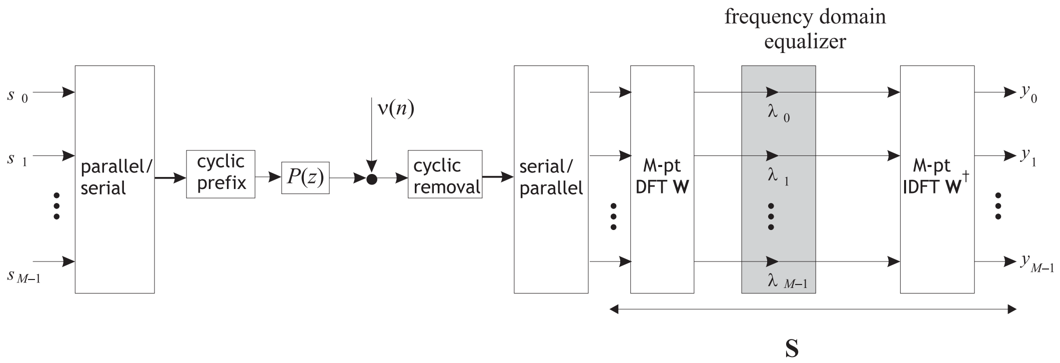

Figure 2: The block diagram of the SC-CP system.

For QPSK modulation with signal variance $\mathcal{E}_s$ and noise variance σ_e^2, the BER is $\mathcal{P} = Q\left(\sqrt{\mathcal{E}_s/\sigma_e^2}\right)$, where $Q(y) = \int_y^\infty e^{-t^2/2}dt/\sqrt{2\pi},\ y \geq 0$. Therefore the k-th subchannel has BER $= Q\left(\sqrt{1/\beta(k)}\right)$. For the convenience of notation, we introduce the function

$$f(y) \stackrel{\triangle}{=} Q(1/\sqrt{y}). \tag{3}$$

The k-th subchannel BER can be expressed as $f(\beta(k))$ and the average BER $\mathcal{P}_{ofdm}$ is,

$$\mathcal{P}_{ofdm} = \frac{1}{M}\sum_{i=0}^{M-1} f\left(\beta(i)\right). \tag{4}$$

Notice that in the OFDM system, the receiver outputs are used directly for symbols detection. An MMSE receiver can be obtained from the zero-forcing receiver by replacing the coefficients $1/P_k$ with $\gamma P_k^*/(1+\gamma|P_k|^2)$. This does not change the unbiased SNR and using an MMSE receiver does not change the BER of the OFDM system.

3. ZERO-FORCING SC-CP SYSTEM

Fig. 2 shows the block diagram of the SC-CP system. The transmitter is simply a serial-to-parallel conversion followed by the insertion of L prefix samples for each block of size M. The receiver consists of a DFT matrix, frequency domain equalizers λ_k and an IDFT matrix as shown in Fig. 2. The receiving matrix $\mathbf{S}$ as indicated in Fig. 2 can be expressed as $\mathbf{S} = \mathbf{W}^\dagger\mathbf{\Lambda}\mathbf{W}$, where $\mathbf{\Lambda}$ is a diagonal matrix with k-th diagonal element λ_k. For a zero-forcing receiver, $\lambda_k = 1/P_k$. Similar to the OFDM system, the system is ISI free as long as the prefix length L is not smaller then the channel order. The overall complexity, equivalent to two DFT matrices plus frequency domain equalizers, is the same as that of the OFDM system.

It is shown in [7] that all the subchannels in the zero-forcing SC-CP system have the same noise variance and it is equal to the average mean squared error $\mathcal{E}_{rr}$ of the OFDM system given in (1). Thus all the subchannel NSRs ($= \mathcal{E}_{rr}/\mathcal{E}_s$) are the same and it is given by $\frac{1}{M}\sum_{i=0}^{M-1}\frac{1}{\gamma|P_i|^2}$. As all the subchannels have identical NSR, all the subchannels have identical BER and the average BER is the same as subchannel BERs. For QPSK symbols, the subchannel BER is $f\left(\mathcal{E}_{rr}/\mathcal{E}_s\right)$ and hence the average BER is given by

$$\mathcal{P}_{sc-cp,zf} = f\left(\frac{1}{M}\sum_{i=0}^{M-1}\frac{1}{\gamma|P_i|^2}\right).$$

In the presence of channel spectral nulls, i.e., $P_{i_0} = 0$ for some i_0, we see that all the subchannel NSRs of the SC-CP system go to infinity. All the subchannels have BER=0.5 no matter how large SNR γ is. We will see in the next section that robustness against spectral null can be significantly improved by using an MMSE receiver.

4. MMSE SC-CP SYSTEM

In this section, we present the MMSE receiver for the SC-CP system. Let the k-th subchannel noise be $q_k = y_k - s_k$, where y_k are the receiver outputs as shown in Fig. 2. To minimize the mean squared error $\sum_{k=0}^{M-1} E[|q_k|^2]$, the receiving matrix $\mathbf{S}$ should be chosen as in the following lemma (see [8] for a proof).

Lemma 1 *Consider the SC-CP transceiver in Fig. 2. Suppose the inputs s_k are QPSK symbols with variance $\mathcal{E}_s$ and the noise is complex Gaussian with variance $\mathcal{N}_0$. The receiving matrix $\mathbf{S}$ that minimizes the mean squared error is*

$$\mathbf{S} = \mathbf{W}^{\dagger}\mathbf{\Lambda}\mathbf{W}, \quad (5)$$

where $\mathbf{\Lambda}$ is a diagonal matrix with diagonal elements λ_k given by

$$\lambda_k = \gamma P_k^*/(1+\gamma|P_k|^2), \quad \text{where } \gamma = \mathcal{E}_s/N_0. \quad (6)$$

Moreover, the subchannel noises have the same variance $\frac{1}{M}\sum_{k=0}^{M-1}\frac{\mathcal{E}_s}{1+\gamma|P_k|^2}$.

The MMSE receiver can be easily obtained from the zero forcing receiver by modifying the value of λ_i in Fig. 2 from $1/P_i$ to that given in (6). As the subchannels have the same noise variance, they also have the same subchannel NSR,

$$\bar{\beta}_{biased} = \frac{1}{M}\sum_{k=0}^{M-1}\frac{1}{1+\gamma|P_k|^2}, \quad (7)$$

which is a biased quantity as the receiver is an MMSE receiver. The unbiased subchannel SNR is, $SNR_{unbiased} = SNR_{biased} - 1$. Let $\bar{\beta} = 1/SNR_{unbiased}$ be the unbiased subchannel NSR. Then we can verify that the unbiased NSR $\bar{\beta}$ is given by

$$\bar{\beta} = \eta(\bar{\beta}_{biased}), \quad \text{where } \eta(y) \stackrel{\triangle}{=} y/(1-y).$$

When an MMSE receiver is used, the system is not ISI free, and the error does not come from channel noise alone. The output noise is a mixture of channel noise and intercarrier interference from other subchannels. However for a reasonably large M, the error can be well-modeled as a Gaussian random variable because of central limit theorem. Gaussian tail renders a very nice approximation of BER. The subchannel BER is well approximated by $f(\bar{\beta}) = f(\eta(\bar{\beta}_{biased}))$. As all the subchannels have the same BER, the average BER $\mathcal{P}_{sc-cp,mmse}$ is equal to the subchannel BER. Let us define

$$h(y) \stackrel{\triangle}{=} f(\eta(y)) = Q(\sqrt{y^{-1}-1}), \quad 0 < y < 1.$$

Then we can conveniently express the BER of the MMSE SC-CP system as

$$\mathcal{P}_{sc-cp,mmse} = h(\bar{\beta}_{biased}).$$

From Lemma 1 we see that, subchannel noise variances are finite even if the channel has spectral nulls. Using the MMSE receiver, the i_0-th frequency domain equalizer $\lambda_{i_0} = 0$ whenever $P_{i_0} = 0$, whereas in the zero-forcing case λ_{i_0} goes to infinity if $P_{i_0} = 0$. The complexity of the MMSE receiver is the same as the zero-forcing receiver, i.e., one DFT matrix, one IDFT matrix and M multipliers λ_k.

5. BER OF MMSE SC-CP AND OFDM SYSTEMS

The function $h(y) = f(\eta(y))$ defined in the previous section can be verified to be a convex function with first and second derivatives respectively satisfying $h'(y) > 0$ and $h''(y) \geq 0$. Using the convexity of $h(\cdot)$, we can show the following theorem.

Theorem 1 *For QPSK modulation symbols, the BERs of the MMSE SC-CP and OFDM systems are related by*

$$\mathcal{P}_{sc-cp,mmse} \leq \mathcal{P}_{ofdm}.$$

The inequality becomes an equality if and only if $|P_0| = |P_1| = \cdots = |P_{M-1}|$.

Proof: Define α_i as

$$\alpha_i = \frac{1}{1+\gamma|P_i|^2}, \quad i = 0, 1, \cdots, M-1.$$

Then the i-th subchannel NSR of the OFDM system in (2) can be written as $\beta(i) = \eta(\alpha_i)$ and the i-th subchannel BER in the OFDM system is $f(\beta(i)) = h(\alpha_i)$. Observe that the biased NSR in (7) can be expressed as $\bar{\beta}_{biased} = 1/M\sum_{i=0}^{M-1}\alpha_i$. Therefore $\mathcal{P}_{sc-cp,mmse} = h(\bar{\beta}_{biased}) = h(1/M\sum_{i=0}^{M-1}\alpha_i)$. Using the facts that $0 < \alpha_i < 1$ and that $h(y)$ is convex for $0 < y < 1$, we have

$$\mathcal{P}_{ofdm} = \frac{1}{M}\sum_{i=0}^{M-1} h(\alpha_i) \geq h\left(\frac{1}{M}\sum_{i=0}^{M-1}\alpha_i\right) = \mathcal{P}_{sc-cp,mmse}.$$

$\triangle\triangle\triangle$

The theorem shows that the SC-CP system has a smaller BER than the OFDM system for all SNR γ.

6. EXAMPLES

We will assume that the noise is AWGN with variance $\mathcal{N}_0$. The modulation symbols are QPSK with values equal to $\pm\sqrt{\mathcal{E}_s/2} \pm j\sqrt{\mathcal{E}_s/2}$ and SNR $\gamma = \mathcal{E}_s/N_0$. The number of subchannels M is 64. The length of cyclic prefix is 3. Two channels with 4 coefficients ($L = 3$) will be used in the examples, $p_1(n) : 0.7768 + j0.4561, -0.0667 + j0.2840, 0.1399 - j0.1592, 0.0223 + j0.2410$ and $p_2(n) : -0.3699 - j0.5782, -0.4053 - j0.5750, -0.0834 - j0.0406, 0.1587 - j0.0156$. The magnitude responses of the two channels $p_1(n)$ and $p_2(n)$ are shown in Fig. 3.

Example 1. We will use $p_1(n)$ in this example. Fig. 4 shows $\mathcal{P}_{ofdm}$ $\mathcal{P}_{sc-cp,zf}$ and $\mathcal{P}_{sc-cp,mmse}$ as functions of SNR γ. The zero-forcing SC-CP system is better than the OFDM system for BER smaller than 10^{-2}. The $\mathcal{P}_{sc-cp,mmse}$ curve is always lower than $\mathcal{P}_{ofdm}$ and $\mathcal{P}_{sc-cp,zf}$. The SC-CP system with an MMSE receiver has a lower BER than the OFDM and zero-forcing SC-CP systems for all SNR.

Example 2. The channel in this example, $p_2(n)$, has a spectral null around 0.9π. The DFT coefficients around 0.9π are very small. Fig. 4 shows the three BER performance curves as in the previous example, $\mathcal{P}_{ofdm}$, $\mathcal{P}_{sc-cp,zf}$ and $\mathcal{P}_{sc-cp,mmse}$. Again $\mathcal{P}_{sc-cp,zf}$ becomes better than $\mathcal{P}_{ofdm}$ for low BER (BER< 0.005). Due to the zero close to the unit circle, the BERs of the two zero forcing systems, $\mathcal{P}_{ofdm}$, and $\mathcal{P}_{sc-cp,zf}$, become small only for large SNR. However there is no serious performance degradation in the SC-CP system with an MMSE receiver.

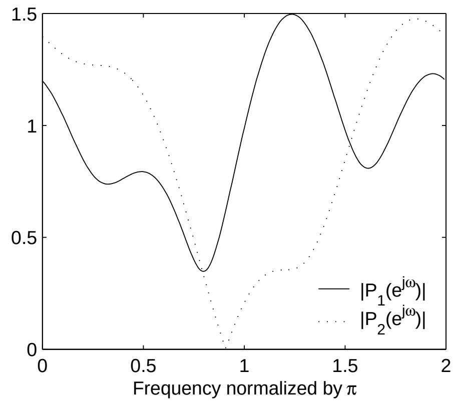

Figure 3: Magnitude responses of the two channels $p_1(n)$ and $p_2(n)$.

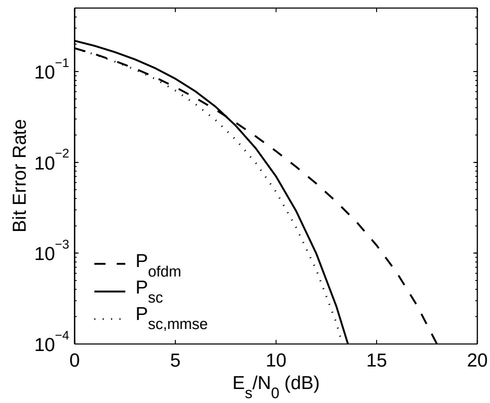

Figure 4: Example 1. Performance comparison of $\mathcal{P}_{ofdm}$, $\mathcal{P}_{sc}$, and $\mathcal{P}_{sc,mmse}$ for the channel $p_1(n)$.

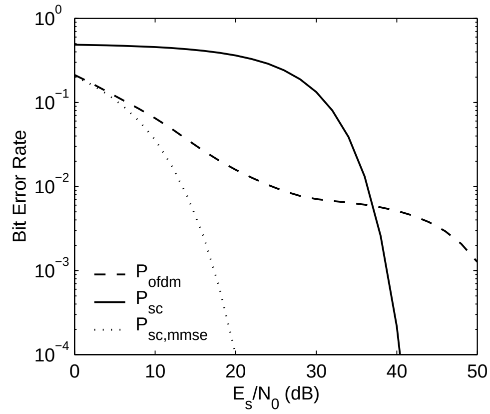

Figure 5: Example 2. Performance comparison of $\mathcal{P}_{ofdm}$, $\mathcal{P}_{sc}$, and $\mathcal{P}_{sc,mmse}$ for the channel $p_2(n)$.

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