MIMO WITH LIMITED FEEDBACK OF CHANNEL STATE INFORMATION

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ABSTRACT

Signaling for optimum mutual information with limited feedback of channel state information is studied for the case when multiple transmit and receive antenna arrays are used to form multiple input multiple output channels. A simple mean square error based approach for finding near optimum signaling is suggested and its performance studied. Prior to limiting, the channel estimates are assumed perfect and delay in obtaining these estimates is ignored to focus on the effects of limited feedback.

1. INTRODUCTION

There continues to be great interest in using multiple transmit and receive antenna arrays to form multiple input multiple output (MIMO) channels for wireless communications [1, 2, 3]. One popular approach [3, 4] employs feedback of channel state information so that the signaling can be tuned to the channel. The optimum approach requires a fairly detailed description of the channel (the eigenvalues and eigenvectors of $\mathbf{H}\mathbf{H}^H$ where $\mathbf{H}$ is the channel matrix) and so it is of interest to study schemes requiring less information. Here we propose an approach that uses $log_2 M$ bits to describe the channel state at a given time where M can be chosen as any integer. Thus the required data rate to feedback the channel state information can be controlled by M and the update rate which is usually dictated by the rate of change of the channel. To focus our efforts on limited M, we assume the unlimited-M channel estimate for $\mathbf{H}$ is still perfect when the signals designed for $\mathbf{H}$ actually arrive at the receiver. This assumes sufficiently slow changes in the channel relative to the rate of signaling.

2. MODEL OF MIMO CHANNEL

Consider an isolated MIMO link with flat Rayleigh fading and additive white Gaussian noise. The vector of complex baseband samples from the set of n_r receive antennas after matched filtering is

$$\mathbf{y} = (y_1, \dots, y_{n_r})^T = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{1}$$

where $\mathbf{x} = (x_1, \dots, x_{n_t})^T$ is the transmitted vector, $\mathbf{H}$ is the channel matrix with independent entries that are each zero-mean complex Gaussian fading coefficients and $\mathbf{n} = (n_1, \dots, n_{n_r})^T$ is the additive zero-mean complex white Gaussian noise vector.

When there is unlimited feedback of channel state information back to the transmitter then the covariance matrix $\mathbf{S}$ can be chosen to maximize

$$\log\left(det(\mathbf{I} + \mathbf{H}\mathbf{S}\mathbf{H}^H)\right) \tag{2}$$

for the given value of $\mathbf{H}$ so that average (ergodic) mutual information is also maximized. Typically $\mathbf{S}$ must also satisfy a constraint on fixed total transmit power. We call the optimum covariance matrix, which maximizes (2) and satisfies the power constraint $\mathbf{S}_{opt}$. This matrix can be found using Lagrange multipliers [3, 4] and a well-known water filling solution results as we now describe. First note that $\mathbf{H}^H\mathbf{H} = \mathbf{U}^H\Lambda\mathbf{U}$ with unitary $\mathbf{U}$ and diagonal $\Lambda = diag(\lambda_1, \dots, \lambda_{n_t})$ due to $\mathbf{H}^H\mathbf{H}$ being Hermitian symmetric. Define an $n_t \times n_t$ diagonal matrix $\tilde{\mathbf{S}}$ with the diagonal entries being

$$\tilde{S}_{jj} = \left(\mu - \frac{1}{\lambda_j}\right)^+, \quad j = 1, \dots, n_t \tag{3}$$

where $(x)^+ = x$ if $x > 0$ and $(x)^+ = 0$ otherwise. In (3), μ is chosen so that the power constraint $\rho = \sum_{j=1}^{n_t} \tilde{S}_{jj}$ is exactly met, where ρ is the largest power allowed. The water filling solution states that the signaling optimizing mutual information employs complex Gaussian signaling with zero mean and a covariance matrix $\mathbf{S}_{opt} = \mathbf{U}^H\tilde{\mathbf{S}}\mathbf{U}$. When this optimum signaling is used, maximum ergodic mutual information can be calculated as

$$E\{\mathcal{I}_F\} = \sum_{j=1}^{n_t} E_{\mathcal{H}}\left\{\left(\log_2(\mu\lambda_j)\right)^+\right\}. \tag{4}$$

where $(x)^+$ in (4) is set to x or 0 in the same way as for the corresponding term (with the same j) in (3).

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Thus we see $\mathbf{S}_{opt}$ is a continuous function of $\mathbf{H}$ so we denote it by $\mathbf{S}_{opt}(\mathbf{H})$ to remind us that this quantity will generally change each time $\mathbf{H}$ changes. Now define the covariance matrix used with the limited feedback operation as $\mathbf{S}_Q(\mathbf{H})$. If we allow only $\log_2 M$ bits of feedback then for each possible value of $\mathbf{H}$ we note that $\mathbf{S}_Q(\mathbf{H})$ can take on one of M different, but fixed, possible values which we denote as $\mathbf{S}_1, \ldots, \mathbf{S}_M$. Describing $\mathbf{S}_Q(\mathbf{H})$ corresponds to describing the M different fixed covariance matrices $\mathbf{S}_1, \ldots, \mathbf{S}_M$ and a mapping from each value of $\mathbf{H}$ to one of these matrices, or equivalently to one of the integers $1, \ldots, M$ (the bits that should be sent as feedback). Thus we define M regions in $\mathbf{H}$ space as $\mathcal{R}_j, j = 1, \ldots, M$ and we make the union of these regions cover the whole space. Further, we coordinate the feedback information and $\mathbf{S}_Q(\mathbf{H})$ so that

$$\mathbf{S}_Q(\mathbf{H}) = \mathbf{S}_j \quad \text{if} \quad \mathbf{H} \in \mathcal{R}_j, \qquad j = 1, \ldots, M \tag{5}$$

We call this a $(log_2 M)$-bit feedback approach.

3. OPTIMUM SIGNALING WITH LIMITED FEEDBACK

We consider finding a mutual information-optimum $(log_2 M)$-bit feedback approach using a two step method. First assume the $\mathbf{S}_1, \ldots, \mathbf{S}_M$ are known. Then maximize instantaneous mutual information (which maximizes ergodic mutual information) by choosing

$$\begin{aligned}\mathcal{R}_j = \{\mathbf{H} :\ & \log\left(det(\mathbf{I} + \mathbf{H}\mathbf{S}_j\mathbf{H}^H)\right) \geq \\ & \log\left(det(\mathbf{I} + \mathbf{H}\mathbf{S}_i\mathbf{H}^H)\right) \quad \forall i \neq j\}.\end{aligned} \tag{6}$$

Of course $\mathbf{H}$ which are on the border (equality in (6)) of two regions $\mathcal{R}_j$ and $\mathcal{R}_i$ $(i \neq j)$ can be assigned to either without changing performance. Let $f_{\mathbf{H}}(\mathbf{H})$ be the probability density function of $\mathbf{H}$. The ergodic mutual information is an integral over all $\mathbf{H}$ as in

$$\begin{aligned}& E\{\mathcal{I}\} = E\{\log\left(det(\mathbf{I} + \mathbf{H}\mathbf{S}_Q(\mathbf{H})\mathbf{H}^H)\right)\} \\ &= \sum_{j=1}^{M} \int \cdots \int_{\mathcal{R}_j} \log\left(det(\mathbf{I} + \mathbf{H}\mathbf{S}_j\mathbf{H}^H)\right) f_{\mathbf{H}}(\mathbf{H}) d\mathbf{H}\end{aligned} \tag{7}$$

and (6) maximizes this integral by the proper choice of $\mathcal{R}_j, j = 1, \ldots, M$.

Next assume the regions $\mathcal{R}_j, j = 1, \ldots, M$ have been defined. Then the optimum $\mathbf{S}_1, \ldots, \mathbf{S}_M$ satisfy

$$\mathbf{S}_j = \arg\max_{\mathbf{S}} \ E\{\log\left(det(\mathbf{I} + \mathbf{H}\mathbf{S}\mathbf{H}^H)\right) | \mathbf{H} \in \mathcal{R}_j\} \tag{8}$$

To see this we note that the ergodic mutual information is given by

$$\begin{aligned}E\{\mathcal{I}\} &= E\{\log\left(det(\mathbf{I} + \mathbf{H}\mathbf{S}_Q(\mathbf{H})\mathbf{H}^H)\right)\} = \\ & \sum_{j=1}^{M} E\{\log\left(\det\left(\mathbf{I} + \mathbf{H}\mathbf{S}_j\mathbf{H}^H\right)\right) | \mathbf{H} \in \mathcal{R}_j\} \\ & \times Prob(\mathbf{H} \in \mathcal{R}_j)\end{aligned} \tag{9}$$

from which (8) clearly follows.

Thus one can search for an optimum signaling approach by iteratively solving (6) and (8). Thus pick an initial guess for $\mathbf{S}_1, \ldots, \mathbf{S}_M$ and then solve (6) for the optimum regions given $\mathbf{S}_1, \ldots, \mathbf{S}_M$. Then update $\mathbf{S}_1, \ldots, \mathbf{S}_M$ from (8) using these regions. Then repeat these steps until convergence. At each step mutual information will always be increased since we are always choosing the updated quantities to optimize mutual information for the fixed other quantities. Thus the resulting procedure must converge since mutual information is limited when we fix signal-to-noise ratio. The resulting solution, upon convergence, must be a so called person-by-person optimum solution meaning that it is a solution that can't be improved by updating just the regions or just the covariance matrices. However in some rare cases it is possible the solution may be improved by updating both at the same time. On the other hand the optimum solution will also be person-by-person optimum thus if you find all person-by-person optimum solutions you will find the optimum solution. In many practical problems only a small number of person-by-person optimum solutions will exist. However, there are some problems where this is not the case. An even greater disadvantage to the method just described is that the maximumization in (8) does not appear to have a simple closed form solution. One would have to solve it numerically [5] which would appear to be a significant disadvantage.

4. MEAN SQUARE ERROR APPROACH

As an alternative approach we propose a procedure that designs the regions and the covariance matrices in order to minimize the mean-squared error (let $||\cdot||_F$ denote the Frobenius norm)

$$\xi = E\{||\mathbf{S}_{opt}(\mathbf{H}) - \mathbf{S}_Q(\mathbf{H})||_F^2\}$$

between $\mathbf{S}_{opt}(\mathbf{H})$ and $\mathbf{S}_Q(\mathbf{H})$. We can expand $\xi =$

$$\begin{aligned}& \int \cdots \int_{\text{all } \mathbf{H}} ||\mathbf{S}_{opt}(\mathbf{H}) - \mathbf{S}_Q(\mathbf{H})||_F^2 \, f_{\mathbf{H}}(\mathbf{H}) d\mathbf{H} = \\ & \sum_{j=1}^{M} \int \cdots \int_{\mathcal{R}_j} ||\mathbf{S}_{opt}(\mathbf{H}) - \mathbf{S}_Q(\mathbf{H})||_F^2 \, f_{\mathbf{H}}(\mathbf{H}) d\mathbf{H} = \\ & \sum_{j=1}^{M} E\{||\mathbf{S}_{opt}(\mathbf{H}) - \mathbf{S}_j||_F^2 \ |\mathbf{H} \in \mathcal{R}_j\} Prob(\mathbf{H} \in \mathcal{R}_j).\end{aligned} \tag{10}$$

The conditional expected value $E\{||\mathbf{S}_{opt}(\mathbf{H}) - \mathbf{S}_j||_F^2 | \mathbf{H} \in \mathcal{R}_j\}$ in (10) can be computed using an integral over the region $\mathcal{R}_j$ as (10) shows.

Now assume $\mathbf{S}_1, \ldots, \mathbf{S}_M$ are given. In order to pick the regions $\mathcal{R}_j, j = 1, \ldots, M$ to minimize ξ we must assign each value of $\mathbf{H}$ to that region which will provide the smallest integrand when evaluated at that value of $\mathbf{H}$. This will make the sum of the integrals in (10) the smallest. Thus

$$\mathcal{R}_j = \{\mathbf{H} : \ ||\mathbf{S}_{opt}(\mathbf{H}) - \mathbf{S}_j||_F^2 < ||\mathbf{S}_{opt}(\mathbf{H}) - \mathbf{S}_i||_F^2, \forall i \neq j\} \quad (11)$$

and $\mathbf{H}$ for which equality occurs in any comparison can be assigned either way (but only to one region) since this does not change our performance measure.

Next assume the regions are given. Then it is easy to show that the optimum $\mathbf{S}_1, \ldots, \mathbf{S}_M$ can be found from

$$\mathbf{S}_j = E\{\mathbf{S}_{opt}(\mathbf{H}) | \mathbf{H} \in \mathcal{R}_j\}, \qquad j = 1, \ldots, M. \quad (12)$$

Again the covariance matrices $\mathbf{S}_1, \ldots, \mathbf{S}_M$ and the regions $\mathcal{R}_j, j = 1, \ldots, M$ are obtained by choosing a starting guess and then iterating using (11) and then (12) repeatedly to get a person-by-person optimum solution. Note that once the $\mathcal{R}_j, j = 1, \ldots, M$ and $\mathbf{S}_1, \ldots, \mathbf{S}_M$ have been designed we can use the designed $\mathbf{S}_1, \ldots, \mathbf{S}_M$ and change $\mathcal{R}_j, j = 1, \ldots, M$ as per (6) to "redefine the regions" to improve mutual information.

5. NUMERICAL RESULTS

Using the procedure just described, the results in Table 1 were obtained. All person-by-person solutions found gave approximately the same performance shown in Table 1. The first column gives signal-to-noise ratio (SNR) in dB. The second column gives ergodic capacity with no feedback (covariance matrix is a scaled identity matrix). The third column gives ergodic capacity with unlimited feedback $M = \infty$. The fourth column gives ergodic mutual information with limited feedback with $M = 4$ using the design approach in (11) and (12), but with the regions redefined after convergence using (6). The fifth column gives the difference between the third and fourth columns which is the ergodic mutual information difference between using feedback with $M = \infty$ and $M = 4$. We note immediately that the ergodic mutual information difference between using feedback with $M = \infty$ and $M = 4$ is nearly a constant over SNR, with a slight decrease in this difference for very small and large SNRs. We conclude that in terms of the size of the error the $M = 4$ approximation is quite good over the full range of possible SNRs. On the other hand some discussion of the relative size of the error compared to the ergodic capacity will more completely describe the exact pattern of the error versus SNR. Thus, if we compare the difference shown in the fifth column of Table 1 with the actual ergodic capacity achieved when $M = \infty$ we see that difference is a very large proportion of the actual ergodic capacity for small SNRs but a very small proportion of the actual ergodic capacity for large SNRs. Thus the relative loss due to limited M is larger for small SNRs.

Table 2 and Table 3 show similar results for cases with $M = 8$ and $M = 16$. Generally the $M = 4$ approach in Table 1 provides ergodic mutual information performance which is quite close to the case with unlimited feedback. As expected the $M = 8$ and $M = 16$ cases provide performance which is even closer to that obtained with $M = \infty$, but the performance of these cases is not that much closer so the extra complexity and larger update communication rate needed for $M = 8$ or $M = 16$ may not be justified. The last column of Table 2 and Table 3 shows the improvement of $M = 16$ over $M = 8$ is even smaller than the improvement of $M = 8$ over $M = 4$. It is also interesting that in each case the improvement is almost constant with SNR but decreases for small and large SNR.

As in Table 1, the difference between the ergodic mutual information with feedback using $M = 8$ or $M = 16$ and unlimited feedback is close to a constant, but the error decreases slightly for small or large SNR. The comments about relative error also apply to Table 2 and Table 3.

6. CONCLUSIONS

Optimum MIMO signaling with limited feedback was studied. A simplified approach based on a mean square error was suggested and its performance studied. Perfect channel estimates were assumed and any delay in obtaining these estimates was ignored.

7. REFERENCES

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Table 1. Ergodic mutual information with various constraints on feedback and design rules. The first column gives signal-to-noise ratio (SNR) in dB. The second column gives ergodic capacity with no feedback (covariance matrix is a scaled identity matrix). The third column gives ergodic capacity with unlimited feedback. The fourth column gives ergodic mutual information with limited feedback with $M = 4$ using the design approach in (11) and (12), but with the regions redefined after convergence using (6). The fifth column shows the loss from using $M = 4$ instead of $M = \infty$.

SNR (dB)	No Feedback	$Feedback M = \infty$	Feedback $M = 4$	$Col_3 - Col_4$
-10	0.2645	0.4249	0.3343	0.0906
-5	0.7220	1.0322	0.8475	0.1847
0	1.6999	2.0991	1.8805	0.2186
5	3.3048	3.6219	3.4288	0.1931
10	5.5640	5.7295	5.6253	0.1042

Table 2. Ergodic mutual information with various constraints on feedback and design rules. The first column gives signal-to-noise ratio (SNR) in dB. The second column gives ergodic capacity with no feedback (covariance matrix is a scaled identity matrix). The third column gives ergodic capacity with unlimited feedback. The fourth column gives ergodic mutual information with limited feedback with $M = 8$ using the design approach in (11) and (12), but with the regions redefined after convergence using (6). The fifth column shows the loss from using $M = 8$ rather than $M = \infty$. The last column shows the loss from using $M = 4$ rather than $M = 8$.

SNR (dB)	No Feedback	Feedback $M = \infty$	Feedback $M = 8$	$Col_3 - Col_4$	$(M = 8) - (M = 4)$
-10	0.2645	0.4249	0.3755	0.0494	0.0412
-5	0.7220	1.0322	0.9386	0.0936	0.0911
0	1.6999	2.0991	1.9650	0.1341	0.0845
5	3.3048	3.6219	3.5116	0.1103	0.0828
10	5.5640	5.7295	5.6484	0.0811	0.0435

Table 3. Ergodic mutual information with various constraints on feedback and design rules. The first column gives signal-to-noise ratio (SNR) in dB. The second column gives ergodic capacity with no feedback (covariance matrix is a scaled identity matrix). The third column gives ergodic capacity with unlimited feedback. The fourth column gives ergodic mutual information with limited feedback with $M = 8$ using the design approach in (11) and (12), but with the regions redefined after convergence using (6). The fifth column shows the loss from using $M = 16$ rather than $M = \infty$. The last column shows the loss from using $M = 8$ rather than $M = 16$.

SNR (dB)	No Feedback	Feedback $M = \infty$	Feedback $M = 16$	$Col_3 - Col_4$	$(M = 16) - (M = 8)$
-10	0.2645	0.4249	0.3942	0.0307	0.0187
-5	0.7220	1.0322	0.9796	0.0526	0.0410
0	1.6999	2.0991	2.0149	0.0842	0.0499
5	3.3048	3.6219	3.5563	0.0656	0.0447
10	5.5640	5.7295	5.6688	0.0607	0.0204