DESIGN OF FULLY-DIVERSE MULTI-ANTENNA CODES BASED ON $SP(2)$

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ABSTRACT

Fully-diverse constellations, i.e., a set of unitary matrices whose pairwise differences are nonsingular, are useful in multi-antenna communications, especially in multi-antenna differential modulation, since they have good pairwise error properties. Recently, group theoretic ideas, especially *£xed-point-free (fpf)* groups, have been used to design fully-diverse constellations of unitary matrices. Here we construct four-transmit-antenna constellations appropriate for differential modulation based on the symplectic group $Sp(2)$. These can be regarded as extensions of Alamouti's celebrated two-transmit-antenna orthogonal design which can be constructed from the group $Sp(1)$. We further show that the structure of the code leads itself to ef£cient maximun likelihood (ML) decoding via the sphere decoding algorithm. Finally, the performance of the code is compared with existing methods including Alamouti's scheme, Cayley differential unitary space-time codes and group based codes.

1. INTRODUCTION

It is well known in theory that multiple antennas can greatly increase the data rate and the reliability of a wireless communication link in a fading environment. In practice, however, one needs to devise effective space-time transmission schemes. This is particularly challenging when the propagation environment is unknown to the sender and the receiver, which is often the case for mobile applications when the channel changes rapidly.

A differential transmission scheme called *differential unitary space-time modulation* was proposed in [1, 2, 3], which is well-tailored for unknown continuously varying Rayleigh ¤at-fading channels. The signals transmitted are unitary matrices. In this scheme the probability of error of mistaking one signal S_i for another $S_{i'}$, at high SNR, is proved to be inversely proportional to $|\det(S_i - S_{i'})|$. Therefore the quality of the code is measured by its *diversity product*

$$\xi_{\mathcal{C}} = \frac{1}{2} \min_{S_i \neq S_{i'} \in \mathcal{C}} |\det(S_i - S_{i'})|^{\frac{1}{M}} \tag{1}$$

where M is the number of transmit antennas and $\mathcal{C}$ is the set of all possible signals. We therefore say that a code is *fully-diverse* or has *full diversity* if the determinants of the pairwise differences are all nonzero. The design problem is thus the following: "Given the number of transmitter antennas, M, and the transmission rate, R, £nd a set $\mathcal{C}$ of $L = 2^{MR}$ $M \times M$ unitary matrices, such that the minimum of the absolute value of the determinant of their pairwise differences is as large as possible."

The design problem, as just stated, appears to be intractable since £rst the signal set and the cost function are non-convex and second, the size of the problem can be huge, especially at high data rates. Therefore, in [4, 5], it was proposed to enforce a group structure on the constellation. This has several advantages that are discussed in [4, 5]. Moreover, it is shown that a constellation is fully-diverse iff the corresponding group is £xed-point-free (fpf), i.e. all non-identity matrices have no eigenvalue at one. In [4], all £nite fully-diverse constellations that form a group are classi£ed. And also, in [5], it is proved that the only fpf in£nite Lie groups are $U(1)$, the group of unit-modulus scalars, and $SU(2)$, the group of unit-determinant 2×2 unitary matrices.

However, no good constellations are obtained for very high rates from the £nite fpf groups classi£ed in [4], and constellations based on $U(1)$ and $SU(2)$ are constrained to one and two-transmit-antenna systems. In this paper, to get high rate constellations which work for 4-transmit-antenna systems, we relax the fpf condition by considering Lie groups with non-identity elements having no more than $k > 0$ unit eigenvalues instead of no unit eigenvalues. It can be shown that if a Lie group has rank n, then it has at least one element with $n-1$ eigenvalues at 1. (The *rank* of a Lie group equals the maximum number of commuting basis elements of its Lie algebra and it can be shown that fpf groups have rank 1. See [5].) The lower the rank, the more possible it is to get a subset with no unit eigenvalue elements, that is, the more possible for us to £nd a fully-diverse subset of

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it. There are only three simply-connected, simple, compact Lie groups of rank 2, the Lie group of unit-determinant 3×3 unitary matrices $SU(3)$, the Lie group of unit-determinant 4×4 unitary, symplectic matrices $Sp(2)$ and one of the exceptional groups $\mathcal{G}_2$. In this paper, we focus on $Sp(2)$. The codes designed based on it are fully-diverse, can be used in four transmit antenna and any number of receive antenna systems, exist for almost any rate and lead themselves to polynomial-time ML decoding via the sphere decoder.

1.1. Differential Unitary Space-time Modulation

Consider a wireless communication system with M transmit antennas and N receive antennas. The channel is used in blocks of M transmissions (for more on this model, see [6, 7]). the system equations of block τ can be written as:

$$X_\tau = \sqrt{\rho} S_\tau H_\tau + V_\tau$$

Here, S denotes the $M \times M$ transmitted signal with s_{tm} the signal sent by the mth transmit antenna at time t. H is the $M \times N$ complex-valued propagation matrix ,which is unknown to both the transmitter and the receiver, and h_{mn} is the propagation coef£cient between the mth transmit antenna and the nth receive antenna and has an iid $\mathcal{CN}(0,1)$ distribution. V is the $M\times N$ noise matrix with v_{tn}, the noise at the nth receive antenna at time t, iid $\mathcal{CN}(0,1)$ distribution. X is the $M \times N$ received signal matrix. The transmitted power constraint is $\sum_{m=1}^{M} \mathrm{E}\,|s_{tm}|^2 = 1, \quad t = 1, ..., M$ so ρ represents the expected SNR at each receive antenna.

In differential modulation, the transmitted matrix S_τ at block τ equals to the product of the previously transmitted matrix and a unitary data matrix V_{z_τ} taken from our signal set $\mathcal{C}$. In other words, $S_\tau = V_{z_\tau} S_{\tau-1}$ where $S_0 = I_M$. The transmission rate is $R = \frac{1}{M}\log_2 L$, where L indicates the cardinality of our code. Further assume that the propagation environment keeps approximately constant for $2M$ consecutive channel uses, that is, $H_\tau \approx H_{\tau-1}$, we may get the fundamental differential receiver equations [8]

$$X_\tau = V_{z_\tau} X_{\tau-1} + W'_\tau \tag{2}$$

where $W'_\tau = W_\tau - V_{z_\tau} W_{\tau-1}$. We can see that the channel matrix H does not appear in (2). This implies that differential transmission permits decoding without knowing the channel information. The ML decoder of z_τ is given by

$$\hat{z}_\tau = \arg\max_{l=0,\ldots,L-1} \|X_\tau - V_l X_{\tau-1}\| \tag{3}$$

It is shown in [1, 3] that, at high SNR, the pairwise probability of error (of transmitting V_l and erroneously decoding $V_{l'}$) has an upper bound that is inversely proportional to the diversity product of the code.

2. MATH FUNDAMENTALS

De£nition 1 (Fixed-point-free Group) *[5] A group $\mathcal{G}$ is called* **£xed-point-free (fpf)** *iff it has a representation as unitary matrices with the property that the representation of each non-unit element of the group has no eigenvalue at unity.*

It can be proved easily that constellations that form a group are fully-diverse iff the group is fpf. In [4], all £nite fpf groups, are classi£ed. These £nite fpf groups are few and far between although there exists an in£nite number of them. Although these yield very good constellations at low to moderate rates, no good constellations are obtained for very high rates from them. This motivates the search for *in£nite* fpf groups, in particular, their most interesting case, Lie groups.

De£nition 2 (Lie Group) *[9] A* **Lie group** *is a differential manifold which is also a group such that the group multiplication and inversion map are differential maps.*

Here are some examples of Lie groups. $GL(n,\mathcal{C})$ is the group of nonsingular $n \times n$ complex matrices. $SL(n,\mathcal{C})$ is the group of unit-determinant nonsingular $n \times n$ complex matrices. $U(n)$ is the group of $n \times n$ complex unitary matrices and $SU(n)$ is the group of unit-determinant $n \times n$ unitary matrices. The following result shows that the groups of interest to us are compact semi-simple Lie groups.

Theorem 1 (Lie groups with Unitary Representations) *[5] A Lie group has a representation as unitary matrices iff it is a compact semi-simple group or the direct sum of $U(1)$ and a compact semi-simple group.*

It is proved in [5], that the only fpf in£nite Lie groups are $U(1)$ and $SU(2)$. Due to their dimensions, constellations based on the two Lie groups are constrained to one and two-transmit-antenna systems. To obtain a four-transmit-antenna constellation, we relax the fpf condition and consider compact semi-simple Lie groups whose non-identity elements have no more than $k > 0$ unit eigenvalues ($k = 0$ corresponds to fpf groups.) In designing a constellation of £nite size, we need to sample the Lie group's underlying manifold. When k is small, there is a good chance that, sampling appropriately, the resulting code is fully-diverse. In general, it does not seem that there is a straightforward way to analyze the number of unit eigenvalues of a matrix element of any given Lie group. However, it is possible to relate the number of unit eigenvalues to the rank of the group. We have proved that if a matrix Lie group $\mathcal{G}$ has rank r, then it has at least one non-identity element with $r-1$ unit eigenvalues. Therefore, instead of exploring Lie groups whose non-identity elements have no more than k unit eigenvalues, we study compact semi-simple Lie groups with rank no more than $k+1$ and design codes that are fully-diverse subsets of it. Since semi-simple Lie groups can be written as a direct product of simple Lie groups, we first consider simple, simply connected, compact Lie groups instead of semi-simple ones with rank 2. As mentioned in the introduction, there are three of them: $SU(3)$, $Sp(2)$ and $\mathcal{G}_2$. Since $Sp(1) = SU(2)$, and $SU(2)$ constitutes the orthogonal design of Alamouti [10], the symplectic group $Sp(2)$ can be regarded as a generalization of orthogonal designs.

Definition 3 (Symplectic Group) *[11] $Sp(n)$, the nth order symptectic group, is the set of complex $2n \times 2n$ matrices S obeying the Unitary condition: $S^*S = SS^* = I_{2n}$ and the symplectic condition: $S^tJS = J$.*

where $J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$, S^t denotes the transpose of S and S^* denotes its conjugate transpose.

$Sp(n)$ has dimension $n(2n+1)$ and rank n. We are most interested in the case of $n = 2$. Actually, it is readily shown that the maximum number of unit eigenvalues of any non-identity element in $Sp(2)$ is 2.

3. $SP(2)$ FULLY-DIVERSE CODE DESIGN

From Definition 3, it is easy to see that any $2n \times 2n$ matrix in $Sp(n)$ has the form $\begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix}$ for some complex $n \times n$ matrices A and B. The group can be identified as the subgroup of unitary matrices with a structure that is similar to Alamouti's 2-dimensional orthogonal design [10], but here each entry is an $n \times n$ matrix instead of a scalar. Using the unitary condition of S and singular value decomposition of A and B, the following theorem can be proved.

Theorem 2 (Parametrization of $Sp(n)$) *Any matrix S belongs to $Sp(n)$ iff it can be written as*

$$S = \begin{bmatrix} U\Sigma_A V & U\Sigma_B \bar{V} \\ -\bar{U}\Sigma_B V & \bar{U}\Sigma_A V \end{bmatrix}$$

where U and V are any $n \times n$ unitary matrices, and $\Sigma_A = \mathrm{diag}\,(\cos\theta_1, \ldots \cos\theta_n), \Sigma_B = \mathrm{diag}\,(\sin\theta_1, \ldots \sin\theta_n)$ *for some real angles $\theta_1, \theta_2 \ldots, \theta_n$. $\bar{U}$ and $\bar{V}$ denote the conjugates of U and V.*

Since any $n \times n$ unitary matrix has dimension n^2, there are all together $2n^2$ degrees of freedom in the unitary matrices U and V. Together with the n real angles, θ_i, the dimension of S is, therefore, $n(2n+1)$, which is exactly the same as that of $Sp(n)$. Based on Theorem 2, the matrices in $Sp(n)$ can be parameterized by U, V and θ_is.

Now, let us look at the easiest case of $n = 2$. For simplicity, we first let $\Sigma_A = \Sigma_B = \frac{1}{\sqrt{2}} I_2$, by which 2 degrees of freedom are lost. We further choose U and V as orthogonal designs with M-PSK and shifted N-PSK entries. The following code is obtained.

$$\mathcal{C}_{M,N} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} UV & U\bar{V} \\ -\bar{U}V & \bar{U}\bar{V} \end{bmatrix} \right\} \quad (4)$$

where $U = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{j\frac{2\pi k}{M}} & e^{j\frac{2\pi l}{M}} \\ -e^{-j\frac{2\pi l}{M}} & e^{-j\frac{2\pi k}{M}} \end{bmatrix}$, and $V = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{j(\frac{2\pi m}{N}+\theta)} & e^{j(\frac{2\pi n}{N}+\theta)} \\ -e^{-j(\frac{2\pi n}{N}+\theta)} & e^{-j(\frac{2\pi m}{N}+\theta)} \end{bmatrix}$ for $0 \le k, l < M, 0 \le m, n < N$ and M and N are integers. θ is an angle to be chosen later. The rate of the code is $\frac{1}{2}(\log_2 M + \log_2 N)$. The angle θ, an extra degree of freedom added to the code to gain diversity product, is crucial in the proof of the full diversity of the code although simulation results indicates that the code always get its highest diversity product at $\theta = 0$.

Since the U and V in our code have an orthogonal design structure, it is not difficult to calculate the determinant of the difference of any two signals in the code directly. Using this calculation, we can prove the following theorem.

Theorem 3 (Condition for full diversity) *There exists a θ such that the code $\mathcal{C}_{M,N}$ in (4) is fully-diverse iff M and N are relatively prime.*

To get codes at higher rates, we can add the two degrees of freedom in diagonal matrices Σ_A and Σ_B in by letting $\Sigma_A = \cos\gamma_i I_2, \Sigma_B = \sin\gamma_i I_2$ for $\gamma_i \in \Gamma$. The full diversity of the modified codes can be proved similarly when θ and the set Γ are properly chosen.

4. DECODING OF THE $SP(2)$ CODE

One of the most prominent properties of our $Sp(2)$ code is that it can be seen as a generalization of orthogonal designs. This property can be used to get linear decoding, which means that the receiver can be made to form a system of linear equations in the unknowns.

From (3), the ML decoder is equivalent to,

$$\arg\max_{U,V} \|X_\tau - \frac{1}{\sqrt{2}} \begin{bmatrix} U & 0 \\ 0 & \bar{U} \end{bmatrix} \begin{bmatrix} I_2 & I_2 \\ -I_2 & I_2 \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & \bar{V} \end{bmatrix} X_{\tau-1}\|_F^2$$
$$= \arg\max_{U,V} \| \begin{bmatrix} U^* & 0 \\ 0 & U^t \end{bmatrix} X_\tau - \frac{1}{\sqrt{2}} \begin{bmatrix} I_2 & I_2 \\ -I_2 & I_2 \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & \bar{V} \end{bmatrix} X_{\tau-1}\|_F^2$$

Note that the formula is quadratic in the entries of U and V. Using the property that U and V constitutes orthogonal designs, it can be shown that the ML decoder reduces to,

$$\arg \max_{0 \le k,l < M, 0 \le m,n < N} \| \begin{bmatrix} \mathcal{A} & -\mathcal{C} \\ \mathcal{B} & -\mathcal{D} \end{bmatrix} \alpha \|_F^2 \quad (5)$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are 4×4 real matrices which only depand on X_τ and $X_{\tau-1}$ and $\alpha = \left[\cos\frac{2\pi k}{M}, \sin\frac{2\pi k}{M}, \cos\frac{2\pi l}{M}, \sin\frac{2\pi l}{M}, \cos\frac{2\pi m}{N}, \sin\frac{2\pi m}{N}, \cos\frac{2\pi n}{N}, \sin\frac{2\pi n}{N}\right]^t$ is the vector of unknowns.

We can see form formula (5) that the decoding criterion is quadratic in the sine and cosine of the unknowns. Thus, it can be solved using the sphere decoder algorithm [12]. By choosing M odd, the map $f : \theta \to \sin\theta$ for $\theta \in \{0, \frac{2\pi}{M}, \cdots, \frac{2(M-1)\pi}{M}\}$ is a one-to-one and onto map. Therefore, we can equivalently regard $\sin\frac{2k\pi}{M}$ and $\sin\frac{2l\pi}{M}$ to be our unknowns instead of k and l. And the same for m and n. Also notice that there are actually 4 independent unknowns instead of 8 in (5). We combine the $2i$-th components (of the form $\cos x$) and the $(2i+1)$-th component (of the form $\sin x$) together in the sphere decoding.

5. SIMULATION RESULTS

In this section, the performance of the $Sp(2)$ code is compared with other codes. The block error rate (bler), which corresponds to errors in decoding the 4×4 transmitted matrices, is demonstrated as the error event of interest.

In Fig 1, we compare our $Sp(2)$ code of $M = 5, N = 3$ and rate $R = 1.95$ with rate 2 orthogonal design and a differential Cayley code at rate 1.75. The number of receive antenna is 1. At a bler of 10^{-3}, the $Sp(2)$ code is 2dB better than the differential Cayley code, even though it has a lower rate, and 4dB better than the orthogonal design.

In Fig 2, we compare our $Sp(2)$ code with a group-based diagonal code and the fpf code $K_{1,1,-1}$ at rate 1.98 [4]. The number of receive antenna is 1. At a bler of 10^{-3}, 2dB improvement is obtained by using the $Sp(2)$ code instead of a diagonal code, but the $Sp(2)$ code is 1.5dB worse than the $K_{1,1,-1}$ group code. However, decoding $K_{1,1,-1}$ requires an exhaustive search over the entire constellation.

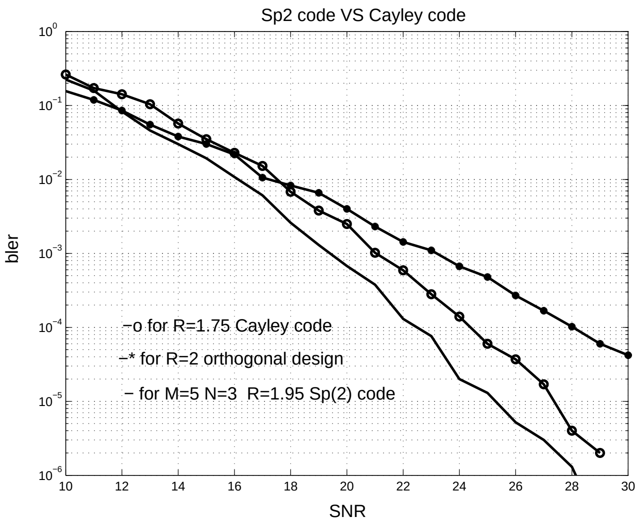

Fig. 1. Performance of $Sp(2)$ code with differential Cayley code and orthogonal design.

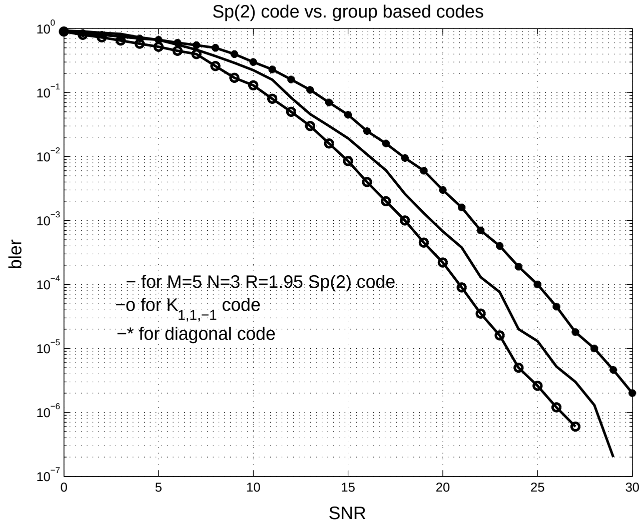

Fig. 2. Performance of $Sp(2)$ code with group-based diagonal code and $K_{1,1,-1}$ code.

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