A SMOOTH EXTENSION FOR THE NONEXPANSIVE ORTHOGONAL WAVELET DECOMPOSITION OF FINITE LENGTH SIGNALS

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ABSTRACT

In transform-based image coding, the periodic extension has the disadvantage that it might introduce high frequency components due to the artificial discontinuities at the signal boundaries. On the other hand, the symmetric extension can't be applied to the two-channel orthogonal filter bank because linear phase is not possible.

This paper describes a smooth extension method, which provides the symmetric decimated outputs of the analysis filters at the boundary, for the two-channel orthogonal filter bank to obtain nonexpansive subband signals. In our approach, extended signal has the flexibility and we calculate the smooth one by singular value decomposition. Finally, several image coding and extended signal examples are shown to confirm the validity of the proposed approach.

1. INTRODUCTION

There has been a significant growth in the field of filter banks and multirate systems [1]. These systems provide efficient ways to represent signals for processing and compression purposes. Wavelets are an even more recent approach in which the two-channel filter bank is iterated to the lowpass branch [2]. Fig. 1 shows the typical structure of the two-channel filter bank. The input signal $x[n]$ is split in two subbands through the analysis filters $H_i(z)$ $(i = 0, 1)$ and decimators. The reconstruction system is formed by interpolators and synthesis filters $F_i(z)$ $(i = 0, 1)$. In general, this system causes the problem that the total number of the subband signals is greater than that of input signal after the analysis process [3]. Given a input $x[n]$ with L samples and filter $h_i[n]$ with M taps, the linear convolution output will have $L + M - 1$ samples. This expansive effect is undesirable in data compression applications. The truncation to obtain nonexpansive subband signals would of course cause distortion in the reconstructed signal.

Several methods for solving this problem have been suggested. One simple nonexpansive approach is based on the signal periodic extension [1]-[3]. Let us consider the example input signal $x[n](n = 0, 1, \ldots, L-1)$ of Fig. 2(a),

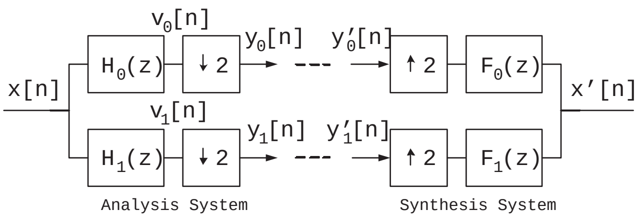

Fig. 1. Two-channel filter bank.

where L is even. The periodic extension means that the input signal is periodically extended as shown in Fig. 2(b) before the signal decomposition, and leads to the periodic outputs $v_i[n]$ with period L for any filters $h_i[n]$. The main drawback with this technique is that the extended signal $\tilde{x}[n]$ might cause large discontinuities across the boundaries and require more bits to code large wavelet coefficients in the highpass band at the boundary.

The symmetric extension method was proposed to overcome the problem of boundary distortion and still obtain a nonexpansive subband signals [3]-[6]. In this approach, input signal is extended periodically and symmetrically as shown in Fig. 2(c) to maintain continuity across the boundaries. Suppose the analysis filter $h_i[n]$ are (anti-)symmetric, the outputs $v_i[n]$ are also (anti-)symmetric and periodic with period $2L$. Then, the half of the samples in signals $v_i[n]$ can be eliminated by symmetry and the output signals $v_i[n]$ need to retain only L samples. However, the symmetric extension has the restriction that filters have to be linear phase. Therefore, this technique can't be applied to the two-channel orthogonal filter bank [1][2].

In this paper, we propose an efficient extension method for nonexpansive orthogonal wavelet decomposition, which make the subband signals symmetric at the boundary. As mentioned above, the subband signals, namely decimated output signals of the analysis bank, based on the periodic extension and the symmetric extension are periodic and/or

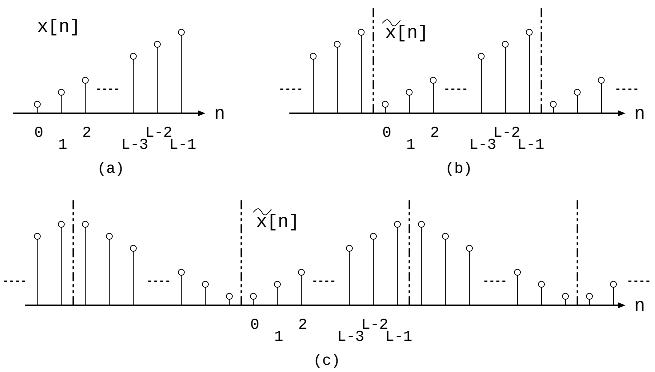

Fig. 2. Illustration of signal extension. (a) The finite length signal. (b) The periodic extension. (c) The symmetric extension.

symmetric. Hence, the coefficient expansion is eliminated and the discarded samples can be retrieved by the priori knowledge before synthesis process. Our scheme is similar to the symmetric extension scheme in that the discarded signal can be retrieved by the symmetry. To achieve smoothness, the extended input samples, which have minimum distance to the closest original input sample, are found by using singular value decomposition. The simulation results for various images show that the proposed extension gives smoother signal and better coding efficiency than the periodic extension, and has the comparable performance to the symmetric extension.

2. A SMOOTH EXTENSION METHOD FOR ORTHOGONAL WAVELET DECOMPOSITION

2.1. Condition of the nonexpansive decomposition

Assuming that a two-channel orthogonal filter bank with $M = 4k + 2j$ tap filters $(k = 1, 2, \ldots; j = 0, 1)$, we will show the extension for the left boundary $(n < 0)$ and the reader can easily infer the extension for the right boundary $(n \geq L)$. In practice, the discarded k samples need to be retrieved before the synthesis system. Let us define that the analysis filter is $h_i[n](i = 0, 1; n = 0, 1, \ldots, M-1)$, the input signal is $x[n](n = 0, 1, \ldots, L-1)$, and the extended signal on the left is $\hat{x}[n](n = -M+1, -M+2, \ldots, -1)$, respectively. Then, the decimated output signals $y_i[n](n = -k, -k+1, \ldots, k-1)$ is given by

$$\mathbf{Y}_i^l = \mathbf{H}_i \mathbf{X}^l, \qquad \text{for} \quad i = 0, 1 \tag{1}$$

$$\mathbf{Y}_i^l = [y_i[-k], \cdots, y_i[-1], y_i[0], \cdots, y_i[k-1]]^T$$

$$\mathbf{X}^l = [\hat{x}[-M+1], \cdots, \hat{x}[-1], x[0] \cdots, x[M-2]]^T$$

$$\mathbf{H}_i = \begin{bmatrix} h_i[M-1] & h_i[M-2] & \cdots & h_i[1] & h_i[0] & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & h_i[M-1] & h_i[M-2] & \cdots & h_i[1] & h_i[0] & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots & 0 & h_i[M-1] & h_i[M-2] & \cdots & h_i[1] & h_i[0] \end{bmatrix}$$

where $\mathbf{Y}_i^l$ is $2k \times 1$ subband vector signal, $\mathbf{X}^l$ is $2(M-1) \times 1$ input vector signal, and $\mathbf{H}_i$ is $2k \times 2(M-1)$ analysis transform matrix created by the circular shifting of the analysis filters $h_i[n]$. Here, the vector $\mathbf{Y}_i^l, \mathbf{X}^l$ are partitioned into two vectors of same length and the matrix $\mathbf{H}_i$ is partitioned into four submatrices of same size as

$$\mathbf{Y}_i^l = \left[\hat{\mathbf{y}}_i^{l\,T}, \mathbf{y}_i^{l\,T}\right]^T \tag{2}$$

$$\mathbf{X}^l = \left[\hat{\mathbf{x}}^{l\,T}, \mathbf{x}^{l\,T}\right]^T \tag{3}$$

$$\mathbf{H}_i = \begin{bmatrix} \mathbf{h}_i^{11} & \mathbf{h}_i^{12} \\ \mathbf{h}_i^{21} & \mathbf{h}_i^{22} \end{bmatrix}. \tag{4}$$

From (2)-(4), the condition for the symmetric subband signals of the analysis system, that is $\mathbf{J}\hat{\mathbf{y}}_i^l = \mathbf{y}_i^l \ (i = 0, 1)$, is given as

$$\mathbf{J}\left(\mathbf{h}_i^{11}\hat{\mathbf{x}}^l + \mathbf{h}_i^{12}\mathbf{x}^l\right) = \mathbf{h}_i^{21}\hat{\mathbf{x}}^l + \mathbf{h}_i^{22}\mathbf{x}^l \qquad \text{for} \quad i = 0, 1 \tag{5}$$

where $\mathbf{J}$ denotes the $k \times k$ reversal matrix. We finally express the condition of the extended input vector $\hat{\mathbf{x}}^l$ in the following way:

$$\mathbf{A}^l \hat{\mathbf{x}}^l = \mathbf{b}^l \tag{6}$$

$$\mathbf{A}^l = \begin{bmatrix} \mathbf{A}_0^l \\ \mathbf{A}_1^l \end{bmatrix} \quad \mathbf{b}^l = \begin{bmatrix} \mathbf{b}_0^l \\ \mathbf{b}_1^l \end{bmatrix} \mathbf{x}^l, \tag{7}$$

where

$$\mathbf{A}_i^l = \mathbf{J}\mathbf{h}_i^{11} - \mathbf{h}_i^{21} \quad \mathbf{b}_i^l = \mathbf{h}_i^{22} - \mathbf{J}\mathbf{h}_i^{12}. \tag{8}$$

2.2. Calculation of the smooth extended signal

From (6)-(8), the extended vector signal $\hat{\mathbf{x}}^l$ is not unique because $\mathbf{A}^l$ is $2k \times (M-1)$ matrix. To restrain discontinuity at the boundaries, we calculate the extended vector signal $\hat{\mathbf{x}}^l$, whose samples $\hat{x}[n] \ (-M+1 \leq n < 0)$ have the minimum distance to the first input sample $x[0]$. This minimization problem for $\hat{\mathbf{x}}^l$ is written as

$$\min \quad \| \hat{\mathbf{x}}^l - x[0]\mathbf{E} \| \qquad \text{subject to} \quad \mathbf{A}^l \hat{\mathbf{x}}^l = \mathbf{b}^l \tag{9}$$

$$\mathbf{E} = [1, 1, \ldots, 1]^T, \tag{10}$$

where $\mathbf{E}$ is $(M-1)\times 1$ vector. Furthermore, if we write $\bar{\mathbf{x}}^l = \hat{\mathbf{x}}^l - x[0]\mathbf{E}$, the equation (9) is rewritten as

$$\min \| \bar{\mathbf{x}}^l \| \quad \text{subject to} \quad \mathbf{A}^l \bar{\mathbf{x}}^l = \mathbf{b}^l - x[0]\mathbf{A}^l\mathbf{E}. \qquad (11)$$

To solve the aforementioned problem, we use the singular value decomposition (SDV). Recall that every matrix has a SVD representation $\mathbf{A} = \mathbf{ULV}^T$, where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices, and $\mathbf{L}$ is a diagonal matrix with the elements w_i. Hence, the inverse matrix $\mathbf{A}^{-1}$ of $\mathbf{A}$ is given as

$$\mathbf{A}^{-1} = \mathbf{V}\,[\,diag(1/w_i)\,]\,\mathbf{U}^T. \qquad (12)$$

Finally, the solution $\bar{\mathbf{x}}^l$ in (11) is given as

$$\bar{\mathbf{x}}^l = \mathbf{V}^l\,[\,diag(1/w_i)\,]^l\,\mathbf{U}^{l\,T} \cdot \left(\mathbf{b}^l - x[0]\mathbf{A}^l\mathbf{E}\right) \qquad (13)$$

and we can calculate the smooth extended vector signal $\hat{\mathbf{x}}^l$ by substituting $\bar{\mathbf{x}}^l$ to $\hat{\mathbf{x}}^l = \bar{\mathbf{x}}^l + x[0]\mathbf{E}$.

3. SIMULATION RESULTS

The performance of the proposed extension is evaluated through an wavelet-based image coding comparison among the proposed extension, the periodic extension, and the symmetric extension. In all cases, five levels decomposition is implemented and the progressive image coder EZW based on intraband partitioning is used[7]. The three images chosen for the image coding experiments are Barbara, Girl, and Lena, which are 512×512 8-bit gray-scale test images.

3.1. Daubechies filter D_4 and $5/3$ filter

We apply the proposed extension and the periodic extension to the minimum-phase Daubechies filter D_4 of length 4, and the symmetric extension to the $5/3$ filter[1][2]. The PSNR's from using three extension methods at various bit rates are tabulated in Table 1. It is clear that the proposed extension technique outperform the periodic extension technique.

3.2. Daubechies filter D_{12} and $9/7$ filter

We apply the proposed extension and the periodic extension to the length 12 Daubechies filter D_{12} with approximate linear phase [8], whose coefficients are shown in Table 2, and the symmetric extension to the $9/7$ filter[1][2]. The PSNR's from using three extension methods at various bit rates are tabulated in Table 3. Extended signal to its left for the 256th row of Barbara image is shown in Fig. 3. It can be clearly observed that the proposed technique introduces smoother extended signal and offers higher coding performance than the periodic extension. Compared with the symmetric extension, the proposed approach has comparable performance.

Table 2. Filter coefficients of the D_{12} filter

n	$h_0[n]$
0	0.01540410932712
1	0.00349071207723
2	-0.11799011119059
3	-0.04831174268055
4	0.49105594184196
5	0.78764114103902
6	0.33792942181793
7	-0.07263752270893
8	-0.02106029248074
9	0.04472490178233
10	0.00176771187070
11	-0.00780070832272

4. CONCLUSIONS

In this paper, we propose an extension approach to process finite length signals via two-channel orthogonal filter banks. To address the expansive problem and guarantee continuity at boundaries, the extended signal is determined in such a way that its output signals are symmetric, and we calculate the smooth one by simple matrix operation. This technique would take little more time to extend input signal in analysis process than the periodic and the symmetric extension. However, there is no difference between the proposed approach and the symmetric approach in synthesis process. The simulation results show that the proposed approach provides a great improvement over the periodic extension and similar performance to the symmetric extension.

5. REFERENCES

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Table 1. Comparisons of PSNR results using various extension method

Image	The proposed extension		The periodic extension		The symmetric extension	
	0.5bpp	1.0bpp	0.5bpp	1.0bpp	0.5bpp	1.0bpp
Barbara	30.36	35.21	30.15	34.98	30.58	35.30
Girl	32.14	35.55	31.71	35.20	32.28	35.48
Lena	35.56	39.13	35.21	38.87	36.08	39.31

Table 3. Comparisons of PSNR results using various extension method

Image	The proposed extension		The periodic extension		The symmetric extension	
	0.5bpp	1.0bpp	0.5bpp	1.0bpp	0.5bpp	1.0bpp
Barbara	32.08	37.29	31.85	37.12	32.00	37.17
Girl	32.69	36.03	32.28	35.68	32.70	35.96
Lena	36.76	40.00	36.29	39.65	36.82	40.02

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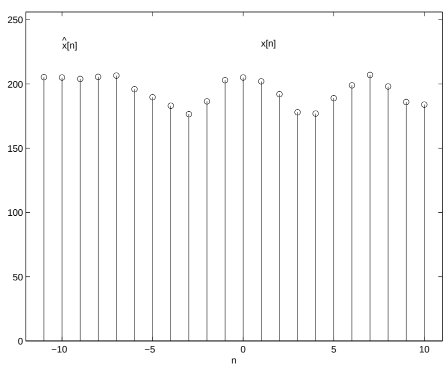
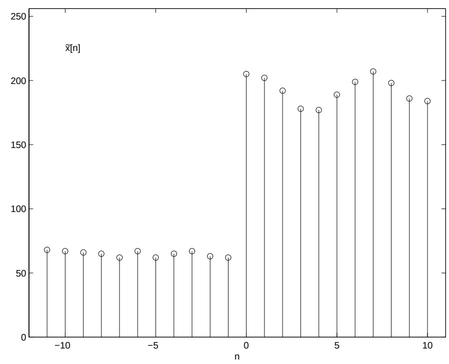
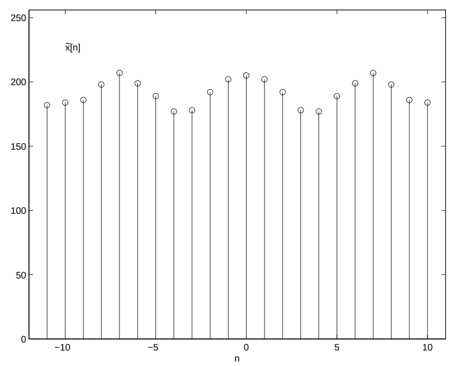


(a) The proposed extension



(b) The periodic extension



(c) The symmetric extension

Fig. 3. Examples of extended signal for 256th row of Barbara image