MULTIDIMENSIONAL SIGNAL PROCESSING USING LOWER-RANK TENSOR APPROXIMATION

Damien MUTI, Salah BOURENNANE

Institut Fresnel / CNRS UMR 6133 - ENSPM,
D.U. de Saint Jérôme F-13397 MARSEILLE Cedex 20, FRANCE.

ABSTRACT

This paper presents the new Fast Optimal Lower Rank Tensor Approximation (FOLRTA) optimal method for lower rank-$(R_1, \ldots, R_N)$ tensor approximation applied for multidimensional signal processing. It is founded on a new approach which consists of considering multidimensional data as global tensors instead of splitting them into matrices or vectors for later classical second order array processing. Its basic principle is to project the initial data tensor into the signal subspace, in each consecutive mode. The developed method is the first analytical solution to the Tucker3 tensor decomposition. We show in a simple example of noise reduction of a color image the efficiency of this method. It can also be applied in Seismic, Acoustics or Multimedia.

1. INTRODUCTION

Multidimensional modelisation of a physical problem can be used in a large range of fields so diverse as chemometric, psychology, data analysis or signal processing [1]. In signal processing, tensors are built on vector spaces associated to physical quantities such as length, width, height, time, color chanel etc... . Each mode of the tensor is associated to a physical quantity. For example, in image processing, color images can be modelled as three-dimensional tensors: two dimensions for lines and columns, and one dimension for the color map. In multimedia signal processing, a sequence of color images can be modelled by a four-dimensional tensor: three modes correspond to color images, and the fourth mode corresponds to the time. In seismic and underwater acoustics a three-dimensional modelisation of data can be adopted as well: one mode is associated to spatial sensors, one mode to the time and one mode to the polarization components of the wave.

So far, most of the processings applied on multidimensional data amount to split these data into observation vectors or matrices in order to apply the classical second order array processing methods. These methods usually involve first or second order statistics, and more recently higher order statistics [2]. Then, the processed vectors or matrices are merged to retrieve the initial tensor.

The splitting of multidimensional data reduces inevitably the quantity of information related to the global tensor as we loose the possibility of studying the relations between components of different slices of data.

In this paper, we propose a new approach in which multidimensional data are considered globally as indivisible tensors in order to potentially dispose of more information than what we could obtain by splitting the data. This new concept leads naturally to use tensorial and multilinear algebra and to elaborate new methods for tensor decomposition and approximation that generalizes the matrix Singular Value Decomposition. We extend to tensor algebra the decomposition of data into two orthogonal subspaces: the noise subspace and the signal subspace. We determine in each mode the noise subspace and the signal subspace, and make the projection of the initial tensor on the signal subspaces. This method is equivalent to find the best lower-rank tensor approximation of the initial tensor.

The developed method is the first analytical solution to the Tucker3 tensor decomposition recently proposed in [3].

In the following, section 2 gives a brief recall of multilinear algebra and tensor decomposition technics. In section 3, we present an original method for multidimensional signal processing based on lower rank-$(R_1, \ldots, R_N)$ tensor approximation. This method consists in finding, in each mode of the tensor, the noise and signal subspaces, taking into account the information contained in all other modes. Finally, in section 4 we apply this new method for the first time to image processing and propose some results for several color images.

2. RECALL OF MULTILIEAR ALGEBRA

In Physics, a tensor of order N can be considered as a multi-dimensional array which entries are accesed via N indexes. We note it $\mathcal{A} \in \Re^{I_1 \times \cdots \times I_N}$, where each element is $a_{i_1 \cdots i_N}$.

The most frequently used model of tensor decomposition is referenced as Tucker3 model [4]. It is the extension of the Singular Value Decomposition in which any N^{th}-order tensor $\mathcal{A} \in \Re^{I_1 \times \cdots \times I_N}$ can be expressed as:

$$\mathcal{A} = \mathcal{S} \times_1 U^{(1)} \times_2 U^{(2)} \cdots \times_N U^{(N)} \quad (1)$$

where $U^{(n)}$ are unitary $I_n \times I_n$-matrice, $\mathcal{S} \in \Re^{I_1 \times \cdots \times I_N}$ is the core tensor and $\times_n$ is the n-mode product, all of which properties can be found in [5]. Given $\mathcal{A} \in \Re^{I_1 \times \cdots \times I_N}$ and a matrix $U \in \Re^{J_n \times I_n}$, $\mathcal{B} = \mathcal{A} \times_n U$ is a tensor of $\Re^{I_1 \times \cdots I_{n-1} \times J_n \times I_{n+1} \times \cdots \times I_N}$, which entries are given by:

$$b_{i_1 \cdots i_{n-1} j_n i_{n+1} \cdots i_N} = \sum_{i_n=1}^{I_n} a_{i_1 \cdots i_{n-1} i_n i_{n+1} \cdots i_N} u_{j_n i_n}. \quad (2)$$

Let's give a brief recall of tensor rank definitions which can be found in [6, 5] and that will be used in the following. The n-rank of tensor $\mathcal{A} \in \Re^{I_1 \times \cdots \times I_N}$, noted as $\mathrm{Rank_k}(\mathcal{A})$, is the dimension of its n-mode vector space E_n composed of the I_n-dimensional vectors obtained from $\mathcal{A}$ by varying index i_n and keeping the other indexes fixed. $\mathcal{A}$ is called a rank-$(R_1, \ldots, R_N)$ tensor if $\mathrm{Rank_k}(\mathcal{A}) = R_k$ whatever $k = 1, \ldots, N$.

Given a real N^{th}-order tensor $\mathcal{A} \in \Re^{I_1 \times \cdots \times I_N}$, the rank-$(R_1, \ldots, R_N)$ tensor approximation problem consists in finding the rank-$(R_1, \ldots, R_N)$ tensor $\mathcal{B}$, with $R_n < I_n$, $\forall n \in [1, N]$, which minimizes, in a Least Square sens, its quadratic Frobenius distance with tensor $\mathcal{A}$: $\|\mathcal{A} - \mathcal{B}\|^2$.

So far, the classical methods for lower-rank tensor approximation have been based on Alternative Least Square (ALS) algorithms and the Higher Order Power Method (HOPM)[7, 5, 8, 6]. They are numerical iterative methods which need expensive computational load and which performances are considerably dependent on the initialization values, the number of iterations and the incremental step value.

Nevertheless, the recently developed Fast Optimal Lower-Rank Tensor Approximation (FOLRTA) algorithm gives an exact analytical solution to rank-$(R_1, \ldots, R_N)$ tensor approximation [3].

3. PROPOSED APPROACH

Let's suppose we dispose of a multidimensional data tensor $\mathcal{A}$, superposition of signal and noise, recorded from physical phenomena. Let E_n be the vector space of dimension I_n, associated to the n-mode of tensor $\mathcal{A}$. E_n is the superposition of two orthogonal subspaces: the signal subspace E_n^1 of dimension R_n, and the noise subspace E_n^2 of dimension $I_n - R_n$, such that $E_n = E_n^1 \oplus E_n^2$.

The goal of this approach is to determine the signal subtensor given by the best rank-$(R_1, \ldots, R_N)$ tensor approximation of $\mathcal{A}$, thanks to consecutive projections of $\mathcal{A}$ on the signal subspaces, in every mode. This can be done thanks to the classical assumption that the signal energy is larger than the noise energy, in every mode.

We present, in a first part, FOLRTA algorithm which gives the best lower-rank tensor approximation. Then, we make a link between this mathematical tool and signal processing. Finally, we give an implementation of this approach when we only dispose of one realisation of the data tensor.

3.1. FOLRTA algorithm

Given any tensor $\mathcal{A} \in \Re^{I_1 \times \cdots \times I_N}$, lower rank-$(R_1, \ldots, R_N)$ tensor approximation of $\mathcal{A}$ consists in finding rank-$(R_1, \ldots, R_N)$ tensor $\mathcal{B} \in \Re^{I_1 \times \cdots \times I_N}$ that minimizes $\|\mathcal{A} - \mathcal{B}\|^2$. According to [5], $\mathcal{B}$ can be expressed with Tucker3 model by:

$$\mathcal{B} = \mathcal{D} \times_1 U^{(1)} \times_2 U^{(2)} \cdots \times_N U^{(N)}, \quad (3)$$

with $U^{(n)} \in \mathrm{St}(I_n, R_n)$ the Stiefel matrix manifold, such that the column vectors of $U^{(n)}$, $n = 1, \ldots, N$, are orthonormal, and $\mathcal{D} \in \Re^{R_1 \times \cdots \times R_N}$ being the core tensor. As given in [5, 8], this problem is equivalent to maximize $\|\mathcal{D}\|^2$, with:

$$\mathcal{D} = \mathcal{A} \times_1 U^{(1)^T} \cdots \times_N U^{(N)^T}, \quad (4)$$

with respect to matrices $U^{(n)} \in \mathrm{St}(I_n, R_n)$, for $n = 1$ to N.

The optimal analytical solution is given by [3]. The column vectors of matrices $U^{(n)}$ are found independently in every n-mode of tensor $\mathcal{A}$. They consist of the R_n eigenvectors associated to the R_n largest eigenvalues of symetric, defined and positive matrix A_n which generic term is:

$$a_{ij}^n = \langle \mathcal{A}_{i_n=i} | \mathcal{A}_{i_n=j} \rangle, \quad (5)$$

where $\mathcal{A}_{i_n=i}$ and $\mathcal{A}_{i_n=j}$ are $(N-1)^{th}$-order tensor from $\Re^{I_1 \times \cdots \times I_{n-1} \times I_{n+1} \times \cdots \times I_N}$ obtained by fixing i_n, the n^{th} mode index of tensor A, respectively to i and j. The tensor scalar product related to Frobenius distance of two complex N^{th}-order tensors $\mathcal{A}$ and $\mathcal{B} \in \Im^{I_1 \times \cdots \times I_N}$ is given by $\langle \mathcal{A} | \mathcal{B} \rangle = \sum_{i_1=1}^{I_1} \cdots \sum_{i_N=1}^{I_N} a_{i_1 \cdots i_N}^* b_{i_1 \cdots i_N}$.

Moreover, the best lower rank-$(R_1, \ldots, R_N)$ tensor approximation of $\mathcal{A}$ is given by:

$$\mathcal{B} = \mathcal{A} \times_1 P^{(1)} \times_2 P^{(2)} \cdots \times_N P^{(N)}, \quad (6)$$

with $P^{(n)} = U^{(n)} U^{(n)^T}$, the orthogonal projector on the subspace generated by the R_n I_n-dimensional column vectors of $U^{(n)}$, in the n-mode initial vector space.

3.2. Link with signal processing

Let's remark that matrix A_n which generic term is given by (5) is the cross-correlation matrix of the $(N-1)^{th}$-order subtensors in a particular n-mode. These subtensors contain the whole information on this n-mode which is comprised in all the other modes.

Moreover, in each mode, $P_n = U^{(n)}U^{(n)^T}$, with $U^{(n)} \in$ St(I_n, R_n) the optimal matrix found in the best lower rank-$(R_1, \dots, R_N)$ tensor approximation of $\mathcal{A}$ is the orthogonal projector from the initial space E_n to the signal space E_n^1.

3.3. Implementation

Let's consider the case when we only dispose of one realisation of the multidimensional data tensor $\mathcal{A}$. The estimation of matrix A_n, which generic term is given by (5), is done as follow:

$$\widehat{a}_{ij}^n = E\left(\langle \mathcal{Y}_i^{\underline{r}} | \mathcal{Y}_j^{\underline{r}} \rangle\right). \tag{7}$$

$\mathcal{Y}_k^{\underline{r}} \in \Re^{Q_1 \times \dots \times Q_{n-1} \times Q_{n+1} \times \dots \times Q_N}$ is a $(N-1)^{th}$-order subtensor extracted from $\mathcal{A}_{i_n=k}$. Its size is $Q_1 \times \dots \times Q_{n-1} \times Q_{n+1} \times \dots \times Q_N$, with $Q_m \leq I_m$ whatever $m \in [1, N] - \{n\}$. Its position into $\mathcal{A}_{i_n=k}$ is defined by vector $\underline{r} = [r_1, \dots, r_N]^T \in \Re^N$, with $r_n \in [1, I_n]$, whatever $n = 1, \dots, N$. $E()$ is the mathematical expectation approximed by:

$$E\left(\langle \mathcal{Y}_i^{\underline{r}} | \mathcal{Y}_j^{\underline{r}} \rangle\right) = \frac{1}{K} \sum_{p=1}^{P} \langle \mathcal{Y}_i^{\underline{r}} | \mathcal{Y}_j^{\underline{r}} \rangle, \tag{8}$$

with P the total number of positions of subtensor $\mathcal{Y}$ chosen within tensor $\mathcal{A}_{i_n=k}$.

4. SIMULATION AND RESULTS

We apply this original FOLRTA algorithm for the first time to image processing. The main goal of the following manipulations is the noise reduction of color images.

In order to validate the method, we first test this algorithm on a simple grey scale image considered as a 2^{nd}-order tensor without noise. Figure 1 shows that the image is totally retrieved by the lower rank tensor approximation.

On figure 2(a), we consider a more elaborated signal composed by a color image $\mathcal{A} \in \Re^{512 \times 384 \times 3}$ of 512 lines, 384 columns and 3 colors. We add to this image a gaussian noise such that the noisy image $\mathcal{B}$, represented figure 2(b), is:

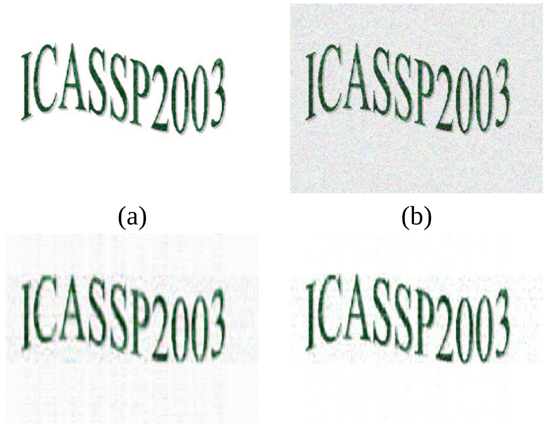
$$\mathcal{B} = \mathcal{A} + 0.2 * 255 * \mathcal{G}, \tag{9}$$

with $\mathcal{G} \in \Re^{512 \times 384 \times 3}$ representing a noise sampled from a three dimensional centered gaussian density of variance 1, and assuming that each color intensity is comprised in $[0, 255]$. In a first approach, we process separately the lower rank-(25,25) approximation of each color chanel (R-G-B) of noisy image figure 2(b) using FOLRTA algorithm, and show the result in image figure 2(c). Then, we process lower rank-(25,25,3) approximation of image 2(b), with FOLRTA algorithm, which result is given figure 2(d). The quality, in terms of noise reduction, of image 2(d) is clearly much better than one of image 2(c), which brievely shows the efficency of considering multidimensional data as global tensors instead of splitting them into matrices or vectors for processings. Moreover, the quality of noise reduction obtained with FOLRTA algorithm is good as most of the noise has disappeared in image 2(d).

Nevertheless, FOLRTA algorithm presents some limitation presented in figure 3. The noise added to the initial image is the same as (9). We process lower rank-(30,30,3) approximation of image 3(b), with FOLRTA algorithm, which result is given figure 3(b). In this case, the final image is slightly destorted and blured by the lower-rank approximation. However, the red and yelow diagonal waves contained

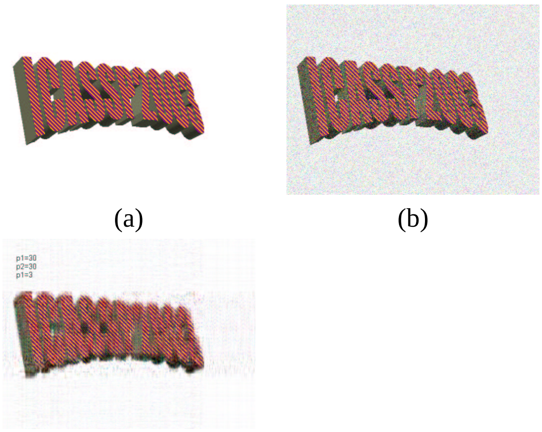
(a) (b)

Fig. 1. Validation of the algorithm on a simple grey scale 153×101 image. (a) initial image without noise. (b) image retrieval after lower-rank tensor approximation.



(a) (b) (c) (d)

Fig. 2. Lower-Rank Tensor Approximation applied on a 512×384 color image. (a) initial color image. (b) noisy image with noise from (eq. 9). (c) color image obtained by processing separately lower rank-(25,25) approximation of each color chanel (R-G-B) of image (b) with FOLRTA algorithm. (d) color image obtained after lower rank-(25,25,3) approximation of image (b), with FOLRTA algorithm.



(a) (b)

(c)

Fig. 3. Lower-Rank Tensor Approximation applied on a 512×384 color image. (a) initial color image. (b) noisy image with noise from (eq. 9). (c) color image obtained after lower rank-(30,30,3) approximation of image (b), with FOLRTA algorithm.

in the image are well retrieved.

5. CONCLUSION

The Fast Optimal Lower-Rank Tensor Approximation is a new original method based on multilinear algebra for multidimensional signal processing, noise reduction or Blind Source Separation. It is founded on a new approach which consists of considering multidimensional data as global tensors instead of splitting them into matrices or vectors for later processing. Its basic principle is to project the initial data tensor on the signal subspace, for each consecutive mode. We show in a simple example of noise reduction of a color image the efficiency of this method.

This novel FOLRTA method can also be applied, not only in image processing, but in Seismic and Acoustics for wave separation, and Multimedia with new perspectives for data compression, for which multidimensional modelisation brings more infomation than what we could get by the classical array processing modelisations and methods. Some more studies are in progress to improve Blind Source Separation especially in the seismic field.

6. REFERENCES

[1] P. Comon, “Tensor decompositions, state of the art and applications,” in *IMA Conf. Mathematics in Signal Processing*, Warwick, UK, Dec. 18-20, 2000, keynote address.

[2] S. Bourennane and A. Bendjama, “Locating wide band acoustic sources using higher order statistics,” *Applied Acoustics*, vol. 63, pp. 235–251, april 2002.

[3] D. Muti and S. Bourennane, “Fast optimal lower-rank tensor approximation,” in *IEEE International Symposium on Signal Processing and Information Technology*, Marrakesh, Morocco, Dec. 18-21, 2002.

[4] L.R. Tucker, “Some mathematical notes on three-mode factor analysis,” *Psychometrika*, vol. 31, pp. 279–311, 1966.

[5] L. De Lathauwer, “On the best rank$(r_1, \ldots, r_N)$,” *SIAM J. Matrix Anal. Appl.*, vol. 21, pp. 1324–1342, april 2000.

[6] E. Kofidis and P.A. Regalia, “On the best rank-1 approximation of higher-order supersymetric tensors,” *SIAM J. Matrix Anal. Appl.*, vol. 23, no. 3, pp. 863–884, march 2002.

[7] P.M. Kroonenberg and J. De Leeuw, “Principal Component Analysis of three-mode data by means of alternating least squares algorithms,” *Psychometrika*, vol. 45, pp. 69–97, 1980.

[8] T. Zhang and G.H. Golub, “Rank-one approximation to high order tensor,” *SIAM J. Matrix Anal. Appl.*, vol. 23, no. 2, pp. 534–550, November 2001.