# TWO-DIMENSIONAL FREQUENCY ESTIMATION USING AUTOCORRELATION PHASE FITTING


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## ABSTRACT

In this paper, the problem of two-dimensional (2-D) frequency estimation of a complex sinusoid embedded in a white Gaussian additive noise is addressed. A new frequency estimator based on a least square plane fitting of the estimated autocorrelation phase of the signal is derived. This algorithm requires a 2-D phase unwrapping step which can be easily done. This algorithm is shown to be unbiased and attains the Cramer Rao bounds for high signal to noise ratio ($SNR > 0dB$).

Accuracy and robustness of this new 2-D frequency estimator are statistically assessed by Monte Carlo simulations. The results obtained show that a good local frequency estimation can be achieved with a very simple algorithm, and a very small amount of points used for the autocorrelation estimation.


## 1. INTRODUCTION

Two-dimensional (2-D) frequency estimation has been widely studied. Among its classical applications, we could mention the use of spectral properties for image segmentation or classification (e.g., [1]). In [2], authors proposed to decompose a texture into a sum of an indeterministic and a deterministic field, which can be characterized by 2-D resonant frequencies. Two-dimensional frequency estimation is also of interest in fields such as sonar and radar. This problem can be achieved using 2-D Fourier transform based methods, but they require large data set and stationarity. These assumptions, which are very restrictive for real life images, reduce the use of such methods. Therefore, short term frequency estimators have to be developed. For this purpose, autoregressive methods, which are said to be of high resolution [3], [4], have been proposed (e.g., [5], [6]). However, for real time applications, it is of great importance to design very simple algorithm.

The problem of 2-D frequency estimation of a complex sinusoid embedded in a white Gaussian additive noise is addressed in this paper through the use of the estimated autocorrelation. By fitting a plane on the estimated autocorrelation phase, we are able to estimate the frequencies. This algorithm requires a 2-D unwrapping stage which can be easily done on the estimated autocorrelation phase.

The paper is organized as follows. In section II, the model of the complex signal and its autocorrelation is presented. The algorithm is then developed in section III. Performances are statistically assessed by Monte Carlo simulations in section IV. Finally, in section V, we present the main conclusions.

## 2. TWO-DIMENSIONAL COMPLEX SIGNAL MODEL

In this work, we assume that the signal to process is a complex sinusoid embedded in complex white Gaussian additive noise given by :

$$s(k,l) = A\exp\left(j2\pi(f_1 k + f_2 l + \theta)\right) + n(k,l), \qquad (1)$$

where :

- $k \in [0..K-1]$ and $l \in [0..L-1]$,
- $f_1$ and $_2$ are the frequencies along the $'k'$ and the $'l'$ directions,
- $\theta$ is the initial phase, uniformly distributed in $[0..2\pi]$,
- $n(k,l)$ is the zero-mean complex white gaussian additive noise, with variance $\sigma_n^2$. In this case, variances of real and imaginary parts of $n(k,l)$ are $\sigma_n^2/2$.
- $A$ is a real constant and will be assumed to be 1 for simplifications, without loss of generality.

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The signal to noise ratio ($SNR$) can be defined as $SNR = 1/\sigma_n^2$. Figure 1 represents an example of the real part of $s(k,l)$ given by eq. (1), with $f_1 = f_2 = 0.05$ and $SNR = 10\log_{10}(\frac{1}{\sigma_n^2}) = 0dB$.

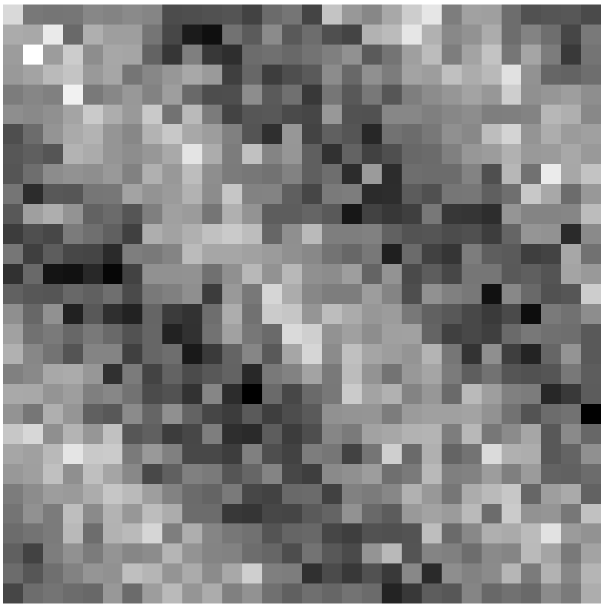

**Fig. 1**. Real part of a $30 \times 30$ signal example for $f_1 = f_2 = 0.05$ and $SNR = 0dB$.

As already mentioned in introduction, we propose in this work a new estimator for $f_1$ and $f_2$ based on the analysis of the unbiased estimation of the autocorrelation sequence of $s(k,l)$ given by :

$$\hat{r}(k,l) = \frac{1}{K-|k|}\frac{1}{L-|l|}\times \sum_{m=0}^{K-|k|-1}\sum_{n=0}^{L-|l|-1} s^*(m,n)s(m+k,n+l). \quad (2)$$

After some straightforward calculations $\hat{r}(k,l)$ can be written in the following form :

$$\hat{r}(k,l) = r_{th}(k,l) + e(k,l), \quad (3)$$

where

- $r_{th}(k,l) = exp(j2\pi(f_1k + f_2l) + \sigma_n^2 d(k,l)$ corresponds to the theoretical autocorrelation sequence,
- $d(k,l)$ is the two-dimensional Dirac function,
- and $e(k,l)$ is the estimation noise of $r(k,l)$.

This complex additive noise can be converted in an additive phase noise, using Tretter's method [7]. For high $SNR$, the autocorrelation estimation variance called $\sigma_e^2$ is low for small values of the lags $(k,l)$. Therefore, eq. (3) can be rewritten as

$$\hat{r}(k,l) = (1 + z(k,l))\exp(j2\pi(f_1k + f_2l)), \quad (4)$$

where $z(k,l) = e(k,l)\exp(-j2\pi(f_1k + f_2l))$. According to Tretter's work [7], when $SNR$ is high, we obtain

$$\begin{aligned} \hat{r}(k,l) = & \left([1 + \textbf{Re}\{z(k,l)\}]^2 + \textbf{Im}\{z(k,l)\}^2\right)^{1/2} \\ & \times \exp\left(j\tan^{-1}\frac{\textbf{Im}\{z(k,l)\}}{1 + \textbf{Re}\{z(k,l)\}}\right) \\ \simeq & \exp(j\textbf{Im}\{z(k,l)\}). \end{aligned} \quad (5)$$

Therefore,

$$\hat{r}(k,l) \simeq \exp(j2\pi(f_1k + f_2l) + j\textbf{Im}\{z(k,l)\}). \quad (6)$$

This result shows that all the required information to estimate $f_1$ and $f_2$ is given by the phase angle of $r(k,l)$. Furthermore the use of the autocorrelation sequence of $s(k,l)$ instead of the signal himself reduces the effect of the additive noise $n(k,l)$, especially when the $SNR$ is high and the autocorrelation lags are small. In the next section, we estimate $f_1$ and $f_2$ using a least square plane fitting of the measured autocorrelation phase.

## 3. FREQUENCY ESTIMATION USING PHASE PLANE FITTING

To estimate $f_1$ and $f_2$, we propose in this work a new algorithm which consists in a least square plane fitting of the phase which has first to be unwrapped. This is usually a quite complicated task in noisy environments and most of the methods already published on this subject try to avoid unwrapping. However, thanks to the use of the autocorrelation, the effect of the noise on the phase is reduced, and the phase can be unwrapped using a very simple algorithm described in the next section. A wrapped phase is depicted in Figure 2 estimated from a $30 \times 30$ image with $f_1 = f_2 = 0.05$ and $SNR = 0dB$.

### 3.1. Phase unwrapping

Several assumptions have to be done on the nature of the autocorrelation phase, in order to unwrap it easily.

1. First of all, the theoretical autocorrelation phase is a plane which equation is

$$\phi(k,l) = w_1k + w_2l, \quad (7)$$

for $|k| < K$ and $|l| < L$.

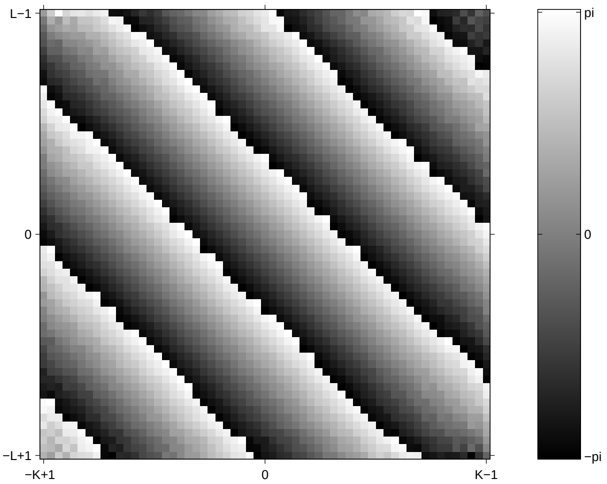


**Fig. 2**. Estimated phase obtained from the autocorrelation of the signal which real part is depicted on figure 1.

2. The estimated autocorrelation is by nature real for $k = l = 0$. Therefore $\hat{\phi}(0,0) = \phi(0,0) = 0$.

3. Shannon's theorem assures that jumps between consecutive theoretical phase of the lattice are lower than $\pi$.

4. The nearer $(0,0)$ is $(k,l)$, the lower is the autocorrelation estimation variance.

Therefore, a very simple algorithm for the 2-D phase unwrapping is to begin from the center of the autocorrelation function i.e. $\phi(0,0)$. The middle column ($k = 0$) is unwrapped from its center, and then, each row is unwrapped from this centered column. Figure 3 shows the unwrapped phase from figure 2.

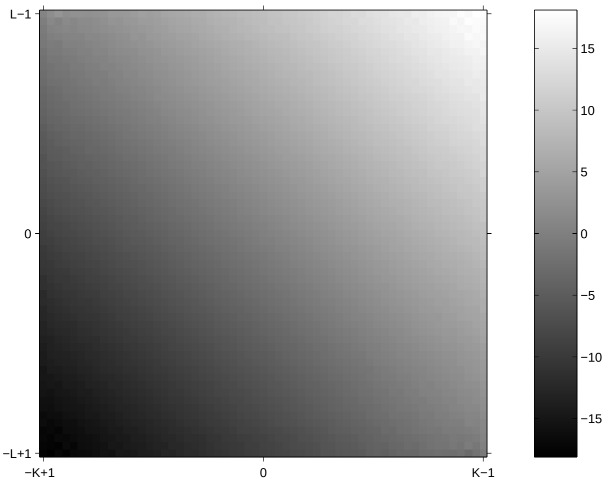


**Fig. 3**. Unwrapped phase from figure 2.

### 3.2. Estimation algorithm

Once the estimated phase is unwrapped, parameters of the expected plane $w_1$ and $w_2$ from eq. (7) have to be estimated. We propose in this work a Least Squares estimation (LS) of those parameters, by plane fitting the estimated autocorrelation phase. This can be done by minimizing the following equation :

$$J(w_1, w_2) = \sum_{k=-M}^{M} \sum_{l=-N}^{N} \left[ w_1 k + w_2 l - \hat{\phi}(k,l) \right]^2, \quad (8)$$

where $[-M..M] \times [-N..N]$ is the lattice used for fitting. It has been previously reported that the autocorrelation estimation variance increases when $|k|$ tends to $K$ or $|l|$ tends to $L$, especially when the $SNR$ is low. As it can be seen in figure 4 for $SNR = -10dB$, phase unwrapping may causes some problems, giving noisy estimations of the frequencies. That is the reason why $M$ and $N$ has to be chosen carefully.

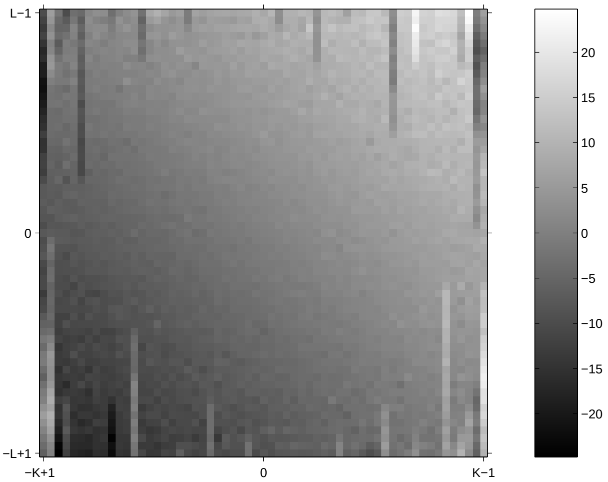


**Fig. 4**. Unwrapped phase from a $-10dB$, $30 \times 30$ image.

By differentiating eq. (8) according to $w_1$ and $w_2$, we find

$$\begin{cases} \sum_{k=-M}^{M} \sum_{l=-N}^{N} k \left[ \hat{w}_1 k + \hat{w}_2 l - \hat{\phi}(k,l) \right] = 0 \\ \sum_{k=-M}^{M} \sum_{l=-N}^{N} l \left[ \hat{w}_1 k + \hat{w}_2 l - \hat{\phi}(k,l) \right] = 0. \end{cases} \quad (9)$$

After simplifications, estimated parameters become

$$\begin{cases} \hat{w}_1 = \dfrac{6}{M(2M+1)(M+1)(2N+1)} \sum_{k=1}^{M} \sum_{l=-N}^{N} k\hat{\phi}(k,l) \\ \hat{w}_2 = \dfrac{6}{N(2N+1)(N+1)(2M+1)} \sum_{l=1}^{N} \sum_{k=-M}^{M} l\hat{\phi}(k,l). \end{cases}$$

Finally, frequencies estimations are given by

$$\hat{f}_1 = \frac{w_1}{2\pi} \quad \text{and} \quad \hat{f}_2 = \frac{w_2}{2\pi}. \tag{10}$$

## 4. PERFORMANCES

In order to study performances of this 2-D frequency estimator, Monte Carlo simulations have been performed. Estimations given by our plane fitting estimator have been compared with the frequency estimator proposed by Kay in [8], for the following parameters :

- $\theta$ uniformly distributed in $[0...2\pi]$,
- $K = L = 30$,
- $f_1 = f_2 = 0.05$,
- $RSB = [-10...30]dB$.

Results obtained show that our estimator has a bias lower than $0.4\%$ for $30 \times 30$ images with $SNR \in [-10...30]dB$, whilst Kay's estimator is about $1\%$ at $5dB$. Variances are depicted in figure 5, and compared with the Cramer Rao bound (CRB).

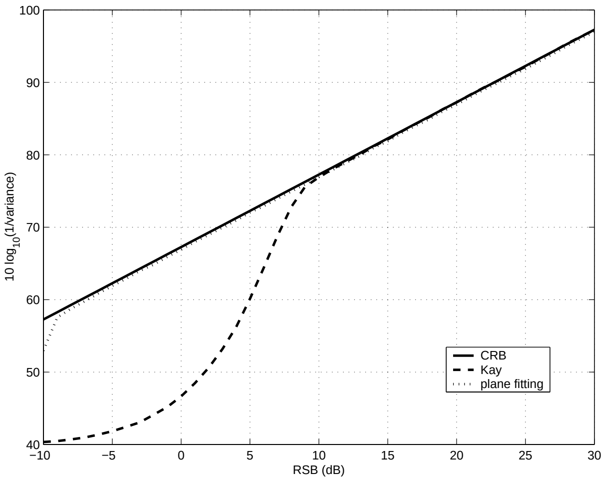


**Fig. 5**. Estimated variances and Cramer Rao bounds.

According to Monte Carlo simulations, the plane fitting frequency estimator appears to be unbiased even for low $SNR$ and its variance reaches the Cramer Rao bound rapidly.

## 5. CONCLUSION

This paper is concerned with the problem of 2-D frequency estimation on complex sinusoid embedded in a white Gaussian additive noise. The proposed algorithm has been derived from the autocorrelation of the complex sinusoid which has a linear phase. The proposed estimator is based on a plane fitting of the estimated autocorrelation phase. Moreover, the algorithm has a low complexity and can be used for real time applications.

Performances of this algorithm have been studied thanks to Monte Carlo simulations in the same conditions used by Kay in [8]. We show the satisfactory behavior of our frequency estimator, which appears to be unbiased and with a variance close to Cramer Rao bounds. This plane fitting algorithm has then been applied to smaller images (16 × 16), giving accurate estimations. These findings suggest that this algorithm can be used for short 2-D term frequency estimation.

## 6. REFERENCES


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