A NEW ALGORITHM FOR RETRIEVAL OF 2D EXPONENTIALS

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ABSTRACT

A novel parametric algorithm that can retrieve roughly 0.25*MN* 2-D exponentials or 0.343*MN* 2-D harmonics from an *M* x *N* array data is presented. This compares favorably with most existing algorithms which can retrieve only order max(*M*, *N*) exponentials or harmonics. The algorithm is not Fourier resolution limited, and requires neither searching in 2-D space nor 2-D polynomial rooting. A specific example of retrieving 4 harmonics from a 3x3 array is developed in detail and numerical performance is demonstrated.

1. INTRODUCTION

Identification and determination of parameters of two-dimensional harmonics are of special interest in signal processing theory. The most prominent application of such is joint azimuth/elevation angle estimation in direction of arrival (DOA) problems using an antenna array. Other applications include joint delay/Doppler estimation in radar signal processing and wireless communications, and edge localization in image processing. Classical non-parametric methods based on 2-D Fourier transform can be effective, efficient and robust if the data sample size is much larger than the number of harmonics to be retrieved. For small size data samples such as typical in array processing, Fourier based methods suffer from resolution limit as well as limits on the number of retrievable harmonics. Many super-resolution techniques have been developed as extensions of 1-D counterparts, including the autoregressive method, maximum entropy method, minimum variance, MUSIC method, Pisarenko's method [1], etc. They either require the search of maxima in 2-D space, or finding roots of a 2-D polynomial, both of which are computationally expensive. There also have been computationally affordable algorithms, each having its own advantages and drawbacks. The state space method cannot handle cases of common 1-D frequencies [2]. The matrix enhancement and matrix pencil method (MEMP) [3] is a superior method in terms of performance and complexity, but it can only retrieve up to $\min(M,N)$ harmonics, where (M,N) are dimensions of the data sample. The algorithm in [4] is formulated for retrieving only one 2-D harmonic.

All of the algorithms mentioned above have the maximum number of retrievable harmonics on the order of $\max(M,N)$ for a $M\times N$ data set. This is considerably smaller than the equations-vs.-unknowns bound, which is $MN/3$ for exponential retrieval and $MN/2$ for harmonic retrieval. Recent works by Jiang and Sidiropoulos [5][6] established that roughly $MN/4$ exponentials (or harmonics) are retrievable in general. Their method is based on low rank decomposition of multi-way arrays. In this paper we present a novel 2-D exponential retrieval algorithm using 1-D polynomial rooting. Our algorithm can retrieve roughly $MN/4$ exponentials, and if the exponentials are restricted to be constant modulus (harmonic retrieval), roughly $0.343MN$ harmonics can be retrieved, which is an improvement over [6].

2. GENERAL FORMULATION FOR CONSISTENT DATA

Let $x[m,n]$, $0\le m\le M-1$, $0\le n\le N-1$ be the (noiseless) complex data which can be modeled as the superposition of R complex exponentials:

$$x[m,n]=\sum_{r=0}^{R-1}c_r z_r^m w_r^n \qquad (1)$$

Here we first consider the general exponential retrieval problem. For harmonic retrieval we would have the additional constraints of $|z_r|=|w_r|=1$. The goal of exponential retrieval is to determine the complex values of $\{c_r,z_r,w_r\}$ given $x[m,n]$. The main difficulty in (1) is the non-linear relationship between the data and $\{z_r,w_r\}$. Inspired by the idea of the Prony's method [7], we assume that the data samples over any $(P+1)\times Q$ sub-array (figure 1) satisfy simultaneously Q linear prediction equations

$$\sum_{p=0}^{P}\sum_{q=0}^{Q-1}\alpha_\gamma[p,q]x[m+p,n+q]=0, \qquad (2)$$

$$0\le m\le M-P-1,\quad 0\le n\le N-Q,\quad 0\le\gamma\le Q-1$$

Then substituting (1) into (2) we obtain Q equations

$$\sum_{p=0}^{P}\sum_{q=0}^{Q-1}\alpha_\gamma[p,q]z^p w^q=0,\quad 0\le\gamma\le Q-1 \qquad (3)$$

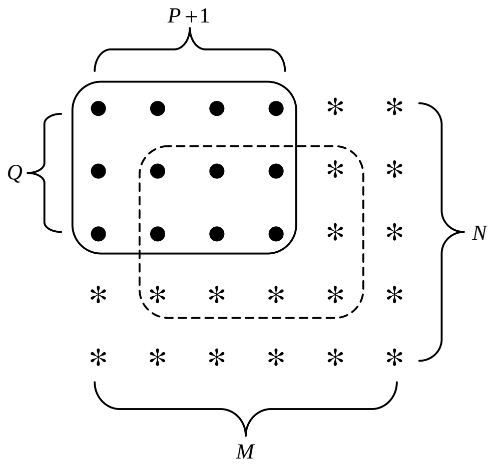

Figure 1: Linear prediction mask

The roots of equation (3) should be $\{z_r, w_r\}$. Note that (3) is a set of over-determined equations because only two equations would be needed to solve for $\{z, w\}$. However, since we have assumed a consistent data set, $\{z_r, w_r\}$ should satisfy all Q equations. Define a $Q \times Q$ matrix polynomial $\mathbf{A}(z)$ whose elements are

$$\left[\mathbf{A}(z)\right]_{\gamma,q} = \sum_{p=0}^{P} \alpha_\gamma[p,q] z^p \qquad (4)$$

then equation (3) can be written as

$$\sum_{q=0}^{Q-1} \left[\mathbf{A}(z)\right]_{\gamma,q} w^q = 0, \quad 0 \le \gamma \le Q-1 \qquad (5)$$

If we treat $1, w, \ldots, w^{Q-1}$ as independent variables (though in fact they are not), then the necessary condition for equation (5) to have a non-trivial solution is

$$\det\left[\mathbf{A}(z)\right] = 0 \qquad (6)$$

Equation (6) is an order PQ polynomial in z. From (6) we can determine $\{z_r\}$ by 1-D polynomial rooting techniques. Subsequently $\{w_r\}$ can be solved through equation (5). Once we get $\{z_r, w_r\}$, the complex amplitudes $\{c_r\}$ can be found by solving the model equation (1), which is linear in $\{c_r\}$.

It remains to find the linear prediction coefficients $\alpha_\gamma[p,q]$. Equation (2) can be written in a matrix form

$$\mathbf{X}\boldsymbol{\alpha}_\gamma = \mathbf{0}$$

where $\mathbf{X}$ is a $(M-P)(N-Q+1) \times (P+1)Q$ matrix and $\boldsymbol{\alpha}_\gamma$ is a $(P+1)Q \times 1$ vector:

$$\mathbf{X} = \begin{bmatrix} x[0,0] & \cdots & x[P,Q-1] \\ \vdots & \ddots & \vdots \\ x[M-P-1,N-Q] & \cdots & x[M-1,N-1] \end{bmatrix} \qquad (7)$$

$$\boldsymbol{\alpha}_\gamma = \begin{bmatrix} \alpha_\gamma[0,0] \\ \vdots \\ \alpha_\gamma[P,Q-1] \end{bmatrix}$$

Therefore $\boldsymbol{\alpha}_\gamma$ is a null space vector of matrix $\mathbf{X}$. We may find all null space vectors of $\mathbf{X}$ through procedures such as the singular value decomposition (SVD) or QR factorization. We must have at least Q null space vectors to form equation (2). This condition determines the maximum number of retrievable exponentials for a given array size, and provides a guideline for choosing the sub-array size P, Q. Since the rank of matrix $\mathbf{X}$ is equal to the number of retrievable exponentials R, we want to find the maximum possible rank of $\mathbf{X}$. Following the matrix rank theorem, we have the inequality $(P+1)Q - R \ge Q$, which leads to

$$R \le PQ \qquad (8)$$

On the other hand the rank of a matrix cannot be bigger than the number of rows, therefore

$$R \le (M-P)(N-Q+1) \qquad (9)$$

The values for P, Q that result in maximum rank R would be such that the right hand sides of the inequality constraints (8) and (9) are about equal. This implies $M = 2P, N = 2Q-1$. For large M, N, $R_{\max} \approx MN/4$, consistent with the results of [5].

If we are restricted to the harmonic retrieval problem where $|z| = |w| = 1$, then the number of rows in the data matrix $\mathbf{X}$ can be doubled by including conjugated and spatial reversed sub-arrays. This is because $z_r^{-1} = z_r^*, w_r^{-1} = w_r^*$. From equation (3) we have

$$\sum_{p=0}^{P}\sum_{q=0}^{Q-1} \alpha_\gamma[p,q] \left(z_r^*\right)^{(M-p)} \left(w_r^*\right)^{(N-q)} = 0 \qquad (10)$$

which in turn leads to the linear prediction equation for the conjugated and spatial reversed data samples:

$$\sum_{p=0}^{P}\sum_{q=0}^{Q-1} \alpha_\gamma[p,q] x^*[M-m-p-1, N-n-q-1] = 0$$

Hence the data matrix in (7) can be augmented to

$$\mathbf{X} = \begin{bmatrix} x[0,0] & \cdots & x[P,Q-1] \\ \vdots & \ddots & \vdots \\ x[M-P-1,N-Q] & \cdots & x[M-1,N-1] \\ x^*[M-1,N-1] & \cdots & x^*[M-P-1,N-Q] \\ \vdots & \ddots & \vdots \\ x^*[P,Q-1] & \cdots & x^*[0,0] \end{bmatrix} \quad (11)$$

Now the inequality in (9) is modified to

$$R \le 2(M-P)(N-Q+1) \quad (12)$$

Equating the right hand side of (8) and (12) gives $P = 0.586M$, $Q = 0.586N$ and $R_{\max} \approx 0.343MN$.

We can summarize our algorithm for retrieving two-dimensional exponentials (harmonics) as follows:

1. Choose $P = 0.5M$, $Q = 0.5N$. Form the data matrix $\mathbf{X}$ according to (7); For harmonic retrievals, choose $P = 0.586M$, $Q = 0.586N$ and form matrix $\mathbf{X}$ according to (11).
2. Obtain Q null space vectors of $\mathbf{X}$, name them $\boldsymbol{\alpha}_\gamma$, $\gamma = 0,1,\ldots,Q-1$;
3. Form the matrix polynomial $\mathbf{A}(z)$ according to (4) and find the PQ roots of $\det[\mathbf{A}(z)] = 0$.
4. For each of the roots z_r solve for a non-zero vector $\mathbf{w}_r$ such that $\mathbf{A}(z_r)\mathbf{w}_r = 0$. It should be possible to normalize $\mathbf{w}_r$ to the form $\mathbf{w}_r = [1, w_r, \ldots, w_r^{Q-1}]^{\mathrm{T}}$, since the data are consistent.
5. Solve for c_r using equation (1).

Note that the actual number of exponentials R determined from the rank of $\mathbf{X}$ is usually smaller than PQ, however this “over-fitting” does not harm the algorithm. Since the data are consistent, simulation shows that the amplitudes of extra exponentials (harmonics) are negligible and can be excluded easily.

3. A CASE STUDY FOR HARMONIC RETRIEVAL FROM 3X3 ARRAY

In this section we illustrate the approach developed in section 2 using a 3 by 3 data set. We consider the problem of harmonic retrieval. According to (8) and (12) we may choose $P = Q = 2$, which implies that up to four 2-D harmonics can be retrieved. The model equation becomes

$$x[m,n] = \sum_{r=0}^{3} c_r z_r^m w_r^n, \quad m = 0,1,2, \quad n = 0,1,2,$$

and the data matrix $\mathbf{X}$ is formed according to (11) as:

$$\mathbf{X} = \begin{bmatrix} x[0,0] & x[0,1] & x[0,2] & x[1,0] & x[1,1] & x[1,2] \\ x[1,0] & x[1,1] & x[1,2] & x[2,0] & x[2,1] & x[2,2] \\ x^*[2,2] & x^*[2,1] & x^*[2,0] & x^*[1,2] & x^*[1,1] & x^*[1,0] \\ x^*[1,2] & x^*[1,1] & x^*[1,0] & x^*[0,2] & x^*[0,1] & x^*[0,0] \end{bmatrix}$$

The linear equations $\mathbf{X}\boldsymbol{\alpha} = \mathbf{0}$ have 4 equations and 6 unknowns, therefore two linearly independent solutions $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ exist. The simultaneous equations for determining $\{z, w\}$ are then

$$\left(\alpha_0[0,0]+\alpha_0[0,1]z+\alpha_0[0,2]z^2\right)+\left(\alpha_0[1,0]+\alpha_0[1,1]z+\alpha_0[1,2]z^2\right)w=0$$
$$\left(\alpha_1[0,0]+\alpha_1[0,1]z+\alpha_1[0,2]z^2\right)+\left(\alpha_1[1,0]+\alpha_1[1,1]z+\alpha_1[1,2]z^2\right)w=0$$

For this equation to have a non-trivial solution, we have

$$\left(\alpha_0[0,0]+\alpha_0[0,1]z+\alpha_0[0,2]z^2\right)\left(\alpha_1[1,0]+\alpha_1[1,1]z+\alpha_1[1,2]z^2\right)$$
$$-\left(\alpha_0[1,0]+\alpha_0[1,1]z+\alpha_0[1,2]z^2\right)\left(\alpha_1[0,0]+\alpha_1[0,1]z+\alpha_1[0,2]z^2\right)=0$$

from which we can obtain four roots z_0, z_1, z_2, z_3. The corresponding four values of w_r are obtained by

$$w_r = -\frac{\alpha_0[0,0]+\alpha_0[0,1]z_r+\alpha_0[0,2]z_r^2}{\alpha_0[1,0]+\alpha_0[1,1]z_r+\alpha_0[1,2]z_r^2}, \quad r = 0,1,2,3$$

Finally, the frequency estimates are obtained by:

$$f_{x,r} = \arg\{z_r\}/(2\pi),\ f_{y,r} = \arg\{w_r\}/(2\pi)$$

Implementing the algorithm described above, we performed numerical simulations by generating four 2-D harmonics with random frequencies and unit magnitudes. Without noise, the algorithm performed as expected, recovering all four harmonics correctly in all cases. We then added white Gaussian noise to the model equation (1) and study the noise performance. The Cramer-Rao bound (CRB) of this frequency estimation problem can be computed rather straightforwardly [3]. Note that the CRB is a function of the frequencies $f_{x,r}$ and $f_{y,r}$, therefore the noise performance is also a function of $f_{x,r}$ and $f_{y,r}$. The following results pertain to a case where the frequencies of the 4 harmonics are fixed at (-0.4, 0.2), (-0.1, -0.1), (0.2, 0.3) and (0.3, -0.4).

We define the signal to noise ratio (SNR) as

$$\mathrm{SNR} = 10\log_{10}\frac{\sum_{r=1}^{4}|c_r|^2}{MN\sigma^2}$$

First we set $\sigma^2 = 0.01, |c_r| = 1$ (SNR=16dB). Figure 2 is a scatter plot of frequency estimates of the 4 harmonics for

100 noise realizations. Also plotted are ellipses associated with the CRB.

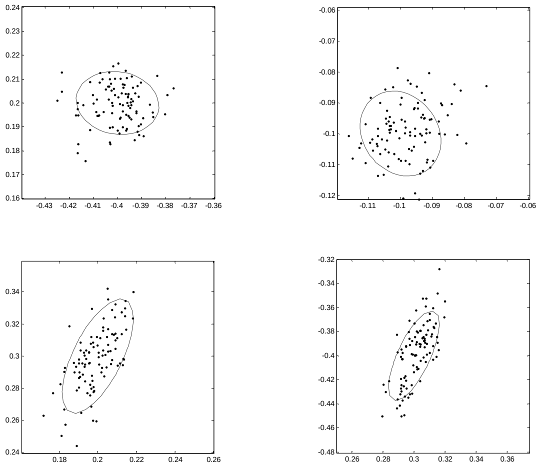

Figure 2: Scatter plot of frequency estimates of all 4 harmonics at 16dB SNR. The ellipses correspond to 2*CRB.

We then varied the SNR from 0 to 35dB, and for each SNR we calculated the mean square error (MSE) of the frequency estimates for 1000 noise realizations. In figure 3, we plot the ratio of MSE/CRB as a function of SNR for all 4 2-D harmonics. It is evident that the algorithm's performance rapidly improves as SNR increases above 20dB. The asymptotic MSE for a frequency component is found to be between 1.2-1.5 times CRB.

4. DISCUSSIONS

As with other parametric algorithms, the algorithm presented in this paper has certain advantages and disadvantages compared to Fourier transform based algorithms: it is not Fourier resolution limited; it can work with small data sets; it is computationally expensive; and it works well for high SNR.

Our algorithm requires rooting of an order PQ 1-D polynomial. To retrieve the maximum number of exponentials (harmonics) given by the equations-vs.-unknowns bound, rooting of 2-D polynomials would be required, since no redundant equations are available as in (3). However, 2-D polynomial rooting is significantly more computationally expensive.

Some extensions of this work could be sought. First, if the number of exponentials to be retrieved is much smaller than the bound of the algorithm $MN/4$, then it is reasonable to reduce complexity by using a smaller (P, Q), in which case the linear prediction equations will be over-determined and a least squares solution should be sought. Second, the technique in this paper could possibly be generalized to higher dimensional data. We are currently investigating these ideas.

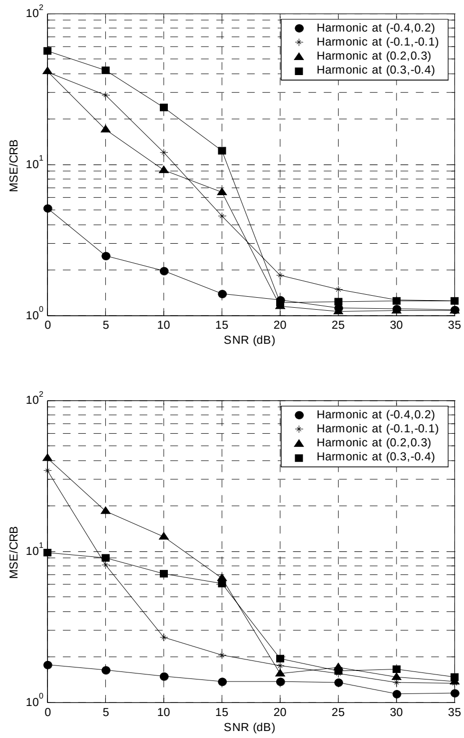

Figure 3: Error performance vs. CRB. Plotted are MSE/CRB vs. SNR for all frequency components along *x* (top) and *y* (bottom).

5. REFERENCES

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