TRANSFORMATION OF LOCAL SPATIO-TEMPORAL STRUCTURE TENSOR FIELDS

Mats Andersson   Hans Knutsson

Dept. of Biomedical Engineering, Linköping University, Sweden
{matsa,knutte}@imt.liu.se

ABSTRACT

Tensors and tensor fields are commonly used in multidimensional signal processing to represent the local structure of the signal. This paper focuses on the case where the sampling on the original signal is anisotropic, e.g when the resolution of the multidimensional image varies depending on the direction which is common e.g. in medical imaging devices. To obtain a geometrically correct description of the local structure there are mainly two possibilities. To resample the image prior to the computation of the local structure tensor field or to compute the tensor field on the original grid and transform the result to obtain a correct geometry of the local structure. This paper deals with the latter alternative and contains an in depth theoretical analysis establishing the appropriate rules for tensor transformations induced by changes in space-time geometry with emphasis on velocity and motion estimation.

1. INTRODUCTION

Local structure tensors[1] [1, 2] has played an important role in multidimensional image processing since the end of the eighties. The concept of tensor representation for local image structure can be generalized to an arbitrary number of dimensions and tensors have e.g been used for representing orientation, velocity, curvature [3], diffusion and are central for adaptive filtering [4] and motion compensation [5].

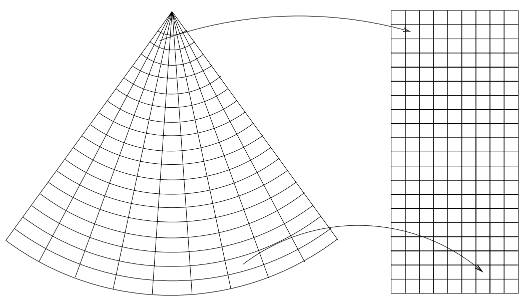

Fig. 1. Actual sampling grid and Initial representation of ultrasound images.

To obtain an true description of the local structure in the signal it is essential that the imaging device provide an isotropic sampling of the signal. For many imaging devices this condition is not fulfilled. In medical imaging CT and MRI volume data do generally have significantly higher resolution within each scan plane compared to the distance between two adjacent planes. Another example is ultrasound images where the scanning is performed on a polar grid, see fig. 1. In order to produce a correct geometry of the local structure such images are generally resampled to a regular grid before the computation of the local structure of the signal. Such a regularization introduce anisotropic properties in the signal as well as in the noise which significantly complicates both the computation of the local structure tensor and subsequent computations such as e.g. estimation of motion fields. For an ultrasound image a resampling to a geometrically correct grid will induce a difference in resolution between the radial and angular directions of more than 5 times in the lower part of the image.

In this paper a new method to overcome the anisotropy effects induced by a resampling is proposed. The local structure tensor field is computed on the *original grid* and is then transformed to coincide with the resampled image with correct geometrical properties. To make such a transformation useful a number of mathematical conditions has to be met but questions of more philosophical art do also appear as e.g should the transformation induce changes in the relations between the eigenvalues of the local tensor. In section 7 an efficient algorithm for transformation of local structure tensors is proposed. In this paper 3D tensors (3D volumes or 2D + time) are used to exemplify the results but the theory is straight forward to generalize to second order tensors of arbitrary dimensionality. In tensor calculus the Einstein summation convention is frequently used to streamline algebraic expressions. For second order tensors a matrix notation is sufficient and as it is expected to be more familiar to most readers this presentation is based on a matrix notation.

2. LOCAL AFFINE COORDINATE TRANSFORMATIONS

Denote the coordinates of the original sample points by the vector $\boldsymbol{x}$. Let $\boldsymbol{A}$ be a local matrix that transforms the coordinate $\mathbf{x}$ to the a geometric correct grid $\boldsymbol{x}'$.

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} \qquad (1)$$

[1] symmetric tensors of order two

The diagonal of $\boldsymbol{A}$ define the scale change in each dimension. When working with MRI and CT data the off diagonal terms are generally zero but for more complex changes in the space-time geometry, as in the ultrasound example, the matrix $\boldsymbol{A}$ also include rotation, shear and deformation. The degrees of freedom in this local affine model of the coordinate transformation is considered to be satisfactory for most sampling patterns. The only formal requirement for the matrix $\boldsymbol{A}$ to be useful as a descriptor of the changes in geometry is that $\det(\boldsymbol{A}) \neq 0$.

3. VELOCITY ESTIMATION FROM LOCAL STRUCTURE TENSORS

There are several methods to compute a local structure tensor $\boldsymbol{T}$. The authors prefer to estimate the local tensor from a set of polar separable quadrature filters [2] but a variety methods exist see e.g.[6]. Independent of how the tensor is estimated a three dimensional tensor of order two can be defined as

$$\boldsymbol{T} = \lambda_1\, \hat{\boldsymbol{e}}_1 \hat{\boldsymbol{e}}_1^T + \lambda_2\, \hat{\boldsymbol{e}}_2 \hat{\boldsymbol{e}}_2^T + \lambda_3\, \hat{\boldsymbol{e}}_3 \hat{\boldsymbol{e}}_3^T \tag{2}$$

where the vectors $[\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3]$ constitute an orthonormal base and the eigenvalues are defined by λ_i. Furthermore it is assumed that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Before the actual tensor transformation is considered lets study the simple example in fig. 2.

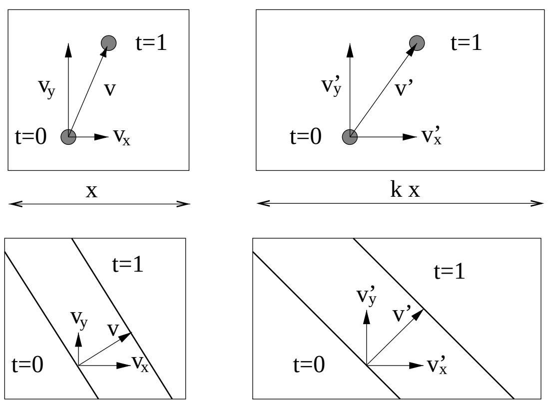

Fig. 2. Scaling along the x-axis, $x' = k\,x$.

The left part of the figure show a point and a line that move a certain distance between $t = 0$ and $t = 1$. The right part of the figure illustrate the same situation where the image is scaled in the horizontal direction. It is apparent that these two velocity estimates behave differently under the transformation. For the moving point, the velocity components are transformed according to the image

$$v_x' = k\, v_x \qquad v_y' = v_y \tag{3}$$

For a moving line or edge only the perpendicular motion component can be estimated locally (the aperture problem). In this case the transformation of the velocity components become more complex.

$$v_x' = k\, v_x\, \frac{v_x^2 + v_y^2}{v_x^2 + k^2 v_y^2} \qquad v_y' = k^2\, v_y\, \frac{v_x^2 + v_y^2}{v_x^2 + k^2 v_y^2} \tag{4}$$

The derivation of eq. (4) is straight forward but cumbersome and is for that reason left out. In this 3D (2D + time) environment a moving line and a moving point corresponds to tensors of different rank which have obvious consequences for the transformation.

4. TRANSFORMATION OF RANK 1 TENSORS

In a sequence of 2D images (2D + time) a moving line will generate a plane. In the Fourier Domain (FD) a plane will only have energy contributions in one direction (perpendicular to the plane) and the resulting tensor will be of rank 1 ($\lambda_2 = \lambda_3 = 0$). From the inclination of the plane the velocity components in the image plane can be deduced. A line moving with the velocity $(v_x, v_y)^T$ corresponds to the following tensor:

$$\boldsymbol{e}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ -v_x^2 - v_y^2 \end{pmatrix} \qquad \boldsymbol{T} = \boldsymbol{e}_1\, \boldsymbol{e}_1^T \tag{5}$$

as for a rank 1 tensor the image velocity is computed as:

$$\boldsymbol{v}_{line} = \frac{-x_3}{x_1^2 + x_2^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{6}$$

see [2, p. 255] for details. Based on the matrix $\boldsymbol{A}$ in eq. (1) there are two possibilities to transform a second order tensor. The transformation can be either ‘covariant’ or ‘contravariant, with respect to the transformation of the grid.

$$\begin{array}{lcll} \boldsymbol{T}' & = & \boldsymbol{A}\,\boldsymbol{T}\,\boldsymbol{A}^T & \text{‘covariant’} \\ \boldsymbol{T}' & = & (\boldsymbol{A}^{-1})^T\,\boldsymbol{T}\,\boldsymbol{A}^{-1} & \text{‘contravariant’} \end{array} \tag{7}$$

As will soon be apparent it is the latter transformation that is relevant in this case. The ‘contravariant’ transformation in eq. (7) can be rewritten as

$$\boldsymbol{T}' = \boldsymbol{e}_1'\, \boldsymbol{e}_1'^{\,T} = (\boldsymbol{A}^{-1})^T \boldsymbol{e}_1\, \left[(\boldsymbol{A}^{-1})^T \boldsymbol{e}_1\right]^T \tag{8}$$

The eigenvector of $\boldsymbol{T}'$ is identified as

$$\boldsymbol{e}_1' = (\boldsymbol{A}^{-1})^T\, \boldsymbol{e}_1 \tag{9}$$

Lets return to the simple example in fig. 2, this case corresponds to an $\boldsymbol{A}$ matrix with $(k, 1, 1)$ in the main diagonal and zeros elsewhere. Using eq. (9) the new eigenvector is computed as

$$\boldsymbol{e}_1' = \begin{pmatrix} k^{-1}\, v_x \\ v_y \\ -(v_x^2 + v_y^2) \end{pmatrix} \tag{10}$$

from which the new image velocity can be computed using eq. (6)

$$\boldsymbol{v}'_{line} = \frac{v_x^2 + v_y^2}{v_x^2 + k^2 v_y^2} \begin{pmatrix} k\, v_x \\ k^2\, v_y \end{pmatrix} \tag{11}$$

which is in agreement with the initial result in eq. (4). Although this is no formal proof we conclude for the moment that the local structure of spatio-temporal rank 1 tensors are preserved when transformed as $\boldsymbol{T}' = (\boldsymbol{A}^{-1})^T \boldsymbol{T} \boldsymbol{A}^{-1}$. Note, however, that norm of the tensor is changed during the transformation. This will be attended to later.

5. TRANSFORMATION OF RANK 2 TENSORS

A moving point in a (2D + time) environment generates a line. In the FD a line has energy contributions in two directions perpendicular to the line. This local structure is represented by a tensor with $\lambda_1 \approx \lambda_2$ and $\lambda_3 = 0$. In this case the velocity information is carried by the third eigenvector. For $\boldsymbol{e}_3 = (x_1, x_2, x_3)^T$ the image velocity is computed as

$$\boldsymbol{v}_{point} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = x_3^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{12}$$

see [2, p. 255]. From the requirements for the moving point example in fig. 2 it is obvious that the transformed rank 2 tensor must fulfill

$$\boldsymbol{e}'_3 = \boldsymbol{A}\, \boldsymbol{e}_3 \qquad \lambda'_3 = 0 \tag{13}$$

Now consider a rank 2 tensor

$$\boldsymbol{T} = \hat{\boldsymbol{e}}_1\, \hat{\boldsymbol{e}}_1^T + \hat{\boldsymbol{e}}_2\, \hat{\boldsymbol{e}}_2^T \tag{14}$$

Let's transform this tensor in accordance with the rank 1 case in the previous section. This may seem as the opposite way but bear with us for a moment.

$$\boldsymbol{T}' = (\boldsymbol{A}^{-1})^T\, \boldsymbol{T}\, \boldsymbol{A}^{-1} \tag{15}$$

The local structure is preserved in this transformation only if

$$\boldsymbol{T}'\, \boldsymbol{A}\, \hat{\boldsymbol{e}}_3 = \boldsymbol{0} \tag{16}$$

To verify this insert eq. (14) and (15) into eq. 16

$$\boldsymbol{T}'\, \boldsymbol{e}'_3 = (\boldsymbol{A}^{-1})^T (\hat{\boldsymbol{e}}_1 \hat{\boldsymbol{e}}_1^T + \hat{\boldsymbol{e}}_2 \hat{\boldsymbol{e}}_2^T)\, \boldsymbol{A}^{-1} \boldsymbol{A}\, \hat{\boldsymbol{e}}_3 = \boldsymbol{0} \tag{17}$$

This somewhat surprising result implies that *both* rank 1 and rank 2 tensors preserves the velocity information when transformed 'Contravariant' with respect to to the image. Note, however that the magnitudes of the nonzero eigenvalues are changed during the transformation.

5.1. A simple rank 2 example

Consider a point moving in the horizontal direction with velocity v. Following the requirements of $\boldsymbol{e}_3$ in eq. (12) this local structure can be expressed as

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ \sqrt{1+v^2} \\ -v \end{pmatrix} \ \boldsymbol{e}_2 = \begin{pmatrix} -1 \\ \sqrt{1+v^2} \\ v \end{pmatrix} \ \boldsymbol{e}_3 = \begin{pmatrix} v \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{T} = \boldsymbol{e}_1\, \boldsymbol{e}_1^T + \boldsymbol{e}_2\, \boldsymbol{e}_2^T$$

Compute $\boldsymbol{T}'$ using the same $\boldsymbol{A}$ matrix as in the rank 1 case.

$$\boldsymbol{T}' = (\boldsymbol{A}^{-1})^T\, \boldsymbol{T}\, \boldsymbol{A}^{-1} = 2 \begin{pmatrix} k^{-2} & 0 & -k^{-1}v \\ 0 & 1+v^2 & 0 \\ -k^{-1}v & 0 & v^2 \end{pmatrix}$$

where $\boldsymbol{e}'_3 = (kv, 0, 1)^T$ and $\boldsymbol{T}'\, \boldsymbol{e}'_3 = \boldsymbol{0}$

6. TRANSFORMATION OF FULL RANK TENSORS

In practice all estimated local structure tensors have rank 3 ($\lambda_3 > 0$). From the distribution of the eigenvalues a decision has to be made on how to interpret the neighborhood when e.g. a velocity field is computed from the local spatio-temporal structure tensors. This is an old problem and the proposed concept offers no solution here but if the original sampling grid is anisotropic it is possible to address the problem one step ahead.

So far we have concluded that the 'contravariant' transformation in eq. (7) preserves the local structure for rank 1 and rank 2 tensors. By inserting a rank 3 tensor into eq. (17) it is obvious that the velocity information is not preserved in the full rank case. Consequently the isotropic part of the tensor has to be removed prior to the transformation.

$$\boldsymbol{T}_0 = \boldsymbol{T} - \lambda_3\, \boldsymbol{I} \tag{18}$$

The eigenvectors of $\boldsymbol{T}$ and the rank 2 tensor $\boldsymbol{T}_0$ are identical and consequently the local structure is preserved in the transformation of $\boldsymbol{T}_0$. The isotropic part can, if desired, be reinserted afterwards. Note that the eigenvalues of $\boldsymbol{T}$ can be computed efficiently e.g by using Cardanos formula [7, 6], without solving the eigenvalue problem. In section 7 an efficient algorithm for transformation of spatio-temporal tensors is presented.

6.1. Mapping of eigenvalues

The relations between the eigenvalues is an important feature for subsequent operations on the tensor fields. Since the relations between the eigenvalues are changed during the transformation this may require some consideration. To simply let the coordinate transformation matrix, $\boldsymbol{A}$, control the entire process may at a cursory glance seem to be the proper way to perform the transformation and ensure that the relation on the new grid is reflected as far as possible by the transformed tensor. For volume data (data with no temporal dimension) this is undoubtly the proper method to perform the transformation. In this case there is no need to limit the rank of the tensor prior to the transformation.

For data set that contain a temporal dimension (spatio-temporal data) additional requirements must be considered as in this case the resulting structure must support a robust estimation of the motion present in the sequence. A change in the relation of the eigenvalues will in this case cause a misinterpretation of the local neighborhood which limits the precision in the estimated motion field. For a time sequence it is consequently better to let the relation between the eigenvalues reflect relations on the original grid while the eigenvectors are transformed. In the next section an efficient algorithm that meets these requirements is presented.

7. AN ALGORITHM FOR TENSOR TRANSFORMATION

This section presents an efficient algorithm for tensor transformation that preserves the original relation of the eigenvalues to enable estimation of the true velocity field on the new grid from the resulting tensor.

1. For the original tensor $\boldsymbol{T}$ compute λ_1,λ_2 and λ_3 e.g. by using Cardanos formula [7, 6].

2. Remove the isotropic part of the tensor, $\boldsymbol{T}_0 = \boldsymbol{T} - \lambda_3\,\boldsymbol{I}$

3. Compute $\boldsymbol{T}'_0 = (\boldsymbol{A}^{-1})^T\,\boldsymbol{T}_0\,\boldsymbol{A}^{-1}$. The transformation preserves the velocity information but the relations between the eigenvalues are changed.

4. Compute λ'_{01} and λ'_{02} for $\boldsymbol{T}'_0$ using Cardanos formula once more ($\lambda'_{03} = 0$ due to step 2).

5. Compute $\boldsymbol{N}_i = e_i e_i^T$ using Knutssons eigenvector formula (for a rank 2 rensor).

$$\begin{array}{rcl} \boldsymbol{N}_1 & = & (\boldsymbol{T}'_0 - \lambda'_{02}\boldsymbol{I})\,\boldsymbol{T}'_0 \\ \boldsymbol{N}_2 & = & (\boldsymbol{T}'_0 - \lambda'_{01}\boldsymbol{I})\,\boldsymbol{T}'_0 \\ \boldsymbol{N}_3 & = & \boldsymbol{I} - \widehat{\boldsymbol{N}}_1 - \widehat{\boldsymbol{N}}_2 \end{array} \tag{19}$$

6. To preserve the original eigenvalues in the transformed tensor compute the resulting tensor as:

$$\boldsymbol{T}' = \lambda_1\widehat{\boldsymbol{N}}_1 + \lambda_2\widehat{\boldsymbol{N}}_2 + \lambda_3\widehat{\boldsymbol{N}}_3 \tag{20}$$

where $\widehat{\ }$ denotes normalization.

Figure 3 shows an example of motion estimation using the above algorithm. The arrows indicate the estimated motion of the heart wall and the mitralis valve.

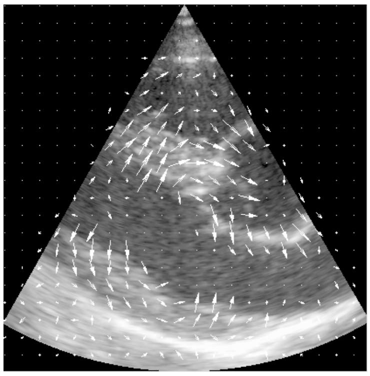

Fig. 3. Motion estimation from ultrasound image.

8. CONCLUSION

The temporal dimension is inherently different from the spatial dimensions when a transformation of the local structure tensor is considered. To preserve the velocity information both the rank and the relations of the eigenvalues must be attended to. In this paper we have presented an efficient algorithm that preserves the velocity information during the transformation which enable estimation of the local structure on the original grid and eliminates the anisotropy effects associated by estimation the local structure after a resampling.

9. ACKNOWLEDGMENT

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