ON BLIND (NON)IDENTIFIABILITY OF DISPERSIVE BANDLIMITED CHANNELS

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ABSTRACT

We study the asymptotic behavior of the smallest singular value of the Single Input Multiple Output (SIMO) channel filtering matrix. We prove that this can be expressed in terms of the sub-channel transfer functions. We apply this result to study the identifiability of bandlimited channels from their (estimated) second order statistics (SOS). We prove, and verify through examples, that SOS based algorithms are unable to identify frequency selective channels regardless of the assumed channel order.

1. INTRODUCTION

Future communication systems tend to involve multi-sensor and fractionally spaced receivers in order to increase the system capacity [9] or allow some tasks, such as SOS-based blind identification, which are not possible with single-sensor Baud-rate receivers. Such techniques lead to the study of Single Input Multiple Output (SIMO) channels. The processing of SIMO channel covariance matrices often involves inversion techniques and blind algorithms show poor performance when tested with bandlimited channels [11]. The interest in the conditioning of such matrices w.r.t. inversion is hence highly justified while few results have been devoted to it [2]. In this paper, we express the conditioning of SIMO channel covariance matrices in terms of the transfer functions of the sub-channels. Application to bandlimited channels is straightforward and enables immediate conclusions on their (non) identifiability from their (estimated) SOS.

2. THE SIMO CHANNEL FILTERING MATRIX

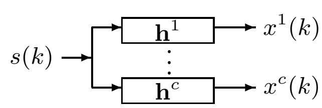


Fig. 1. Single input multiple output channel

An m-order SIMO channel, as depicted in Fig. 1, is a set of c filters $\mathbf{h}^u \hat{=} [h_0^u, \cdots, h_m^u]^T$ (to each we associate the Fourier transform $h^u(w)$), $u = 1, \cdots, c$, driven by a common scalar input $s(k)$. This setting corresponds to a multi-sensor reception or a poly-phase representation of an over-sampled signal, or a possibly hybrid situation. The SIMO channel order m is defined as the maximum order among those of the different filters $\mathbf{h}^1 \cdots \mathbf{h}^c$.

n successive output observations are stacked into the vector $\mathbf{x}_n(k) \hat{=} \left[x^1(k) \cdots x^1(k-(n-1)) \cdots x^c(k) \cdots x^c(k-(n-1))\right]^T$ related to the scalar input by $\mathbf{x}_n(k) = \mathbf{H}_n(\mathbf{h})\mathbf{s}_{n+m}(k)$ where $\mathbf{H}_n(\mathbf{h}) \hat{=} \begin{bmatrix} \mathbf{H}_n(\mathbf{h}^1) \\ \vdots \\ \mathbf{H}_n(\mathbf{h}^c) \end{bmatrix}$ is the $cn \times (n+m)$ SIMO channel *filtering matrix* and $\mathbf{H}_n(\mathbf{h}^u) \hat{=} \begin{bmatrix} \mathbf{h}^{u^T} & 0 & \cdots & 0 \\ 0 & \mathbf{h}^{u^T} & \cdots & 0 \\ & & \vdots & \\ 0 & \cdots & 0 & \mathbf{h}^{u^T} \end{bmatrix}$ is the $n \times (n+m)$ filtering matrix associated with the u-th filter. We note $\sigma_k^{(n)} \hat{=} \sigma_k(\mathbf{H}_n(\mathbf{h}))$ where $\sigma_k(\mathbf{A})$ (resp. $\lambda_k(\mathbf{A})$) refers to the k-th largest singular (resp. eigen) value of the matrix (resp. square matrix) $\mathbf{A}$.

If the input $s(k)$ is zero mean and white with variance σ_s^2, then the covariance matrix $\mathbf{R}_n \hat{=} \mathrm{E}\left[\mathbf{x}_n(k)\mathbf{x}_n^H(k)\right] = \sigma_s^2 \mathbf{H}_n(\mathbf{h})\mathbf{H}_n^H(\mathbf{h})$.

An important result [10] states that the filtering matrix is full column rank if the SIMO channel is zero-coprime (i.e. $h^u(w), u = 1, \cdots, c$ do not have any zero in common) and if $n \geq m$.

3. ASYMPTOTIC BEHAVIOR OF THE FILTERING MATRIX SINGULAR VALUES

We first recall the following definitions and results about asymptotic equivalence [4] as well as Szegö's theorem [5]. The strong norm $\|\mathbf{A}_n\|$ and the weak norm $|\mathbf{A}_n|$ are defined, respectively, as the spectral norm $\|\mathbf{A}_n\|^2 \hat{=} \max_{\|\mathbf{x}\|=1} \mathbf{x}^H \mathbf{A}_n^H \mathbf{A}_n \mathbf{x}$ and as the normalized Frobenius norm $|\mathbf{A}_n|^2 \hat{=} \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n |a_{i,j}|^2$.

Definition 1 *Asymptotic equivalence*
Two $n \times n$ matrix sequences $\{\mathbf{A}_n\}$ and $\{\mathbf{B}_n\}$, $n = 1, 2, \cdots$ are said to be asymptotically equivalent and noted $\{\mathbf{A}_n\} \sim \{\mathbf{B}_n\}$ if

$$\exists M < \infty \text{ such that } \forall n, \|\mathbf{A}_n\| \leq M \text{ and } \|\mathbf{B}_n\| \leq M \quad (1)$$

$$\lim_{n\to\infty} |\mathbf{A}_n - \mathbf{B}_n| = 0 \quad (2)$$

Lemma 1 *If $\{\mathbf{A}_n\} \sim \{\mathbf{B}_n\}$ and if $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k^s(\mathbf{A}_n)$ exists and is finite for any positive integer s, then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k^s(\mathbf{A}_n) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k^s(\mathbf{B}_n)$$

Theorem 1 *Szegö's theorem*
For all absolutely summable sequences $\{t_k\}_{k=\cdots,-1,0,1,\cdots}$, if $\mathbf{T}_n \hat{=} \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{-(n-1)} \\ t_1 & \ddots & \ddots & t_{-(n-2)} \\ \vdots & \ddots & & \vdots \\ t_{n-1} & t_{n-2} & \cdots & t_0 \end{bmatrix}$ *is Hermitian,*

then for all functions F continuous on $[\min_w t(w), \max_w t(w)]$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}F\left(\lambda_k(\mathbf{T}_n)\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}F\left(t(w)\right)\,dw.$$

where $t(w)\hat{=}\sum_k t_k e^{-ikw}$ stands for the Fourier transform of t_k.

In the following, we prove that the sequence $\mathbf{H}_n^H(\mathbf{h})\mathbf{H}_n(\mathbf{h})$ is asymptotically equivalent to a Toeplitz matrix sequence, then, using Szegö's theorem, a result is estalished on the aymptotic behavior of the singular values of $\mathbf{H}_n(\mathbf{h})$.

Let $i,j\in\{1,\ldots,n+m\}$.
$\left(\mathbf{H}_n^H(\mathbf{h}^u)\mathbf{H}_n(\mathbf{h}^u)\right)_{i,j}=\sum_{k=1}^{n}\left(h^u(i-k)\right)^* h^u(j-k)$
$=\sum_{k\in\{1,\ldots,n\}\bigcap\{i-m,\ldots,i\}\bigcap\{j-m,\ldots,j\}}\left(h^u(i-k)\right)^* h^u(j-k)$.
If $j-i>m$ then $\{i-m,\ldots,i\}\bigcap\{j-m,\ldots,j\}=\emptyset$ and $\left(\mathbf{H}_n^H(\mathbf{h}^u)\mathbf{H}_n(\mathbf{h}^u)\right)_{i,j}=0$. If $m+1\le i\le n$ then $\{1,\ldots,n\}\bigcap\{i-m,\ldots,i\}=\{i-m,\ldots,i\}$ and $\left(\mathbf{H}_n^H(\mathbf{h}^u)\mathbf{H}_n(\mathbf{h}^u)\right)_{i,j}$
$=\sum_{k\in\{i-m,\ldots,i\}\bigcap\{j-m,\ldots,j\}}h^{u^*}(i-k)h^u(j-k)$ which is a function of $i-j$. The same holds when $m+1\le j\le l$.

We hence can write $\mathbf{H}_n^H(\mathbf{h}^u)\mathbf{H}_n(\mathbf{h}^u)=$

$$\left[\begin{array}{ccccccccccc}
 & & & t_m^{u^*} & 0 & \cdots & & & & & \\
 & \mathbf{X}_m^u & & \vdots & \ddots & \ddots & & & & & \\
 & & & t_1^{u^*} & \cdots & t_m^{u^*} & & & & & \\
t_m^u & \cdots & t_1^u & t_0^u & t_1^{u^*} & \cdots & t_m^{u^*} & & & & \\
0 & \ddots & \ddots & t_1^u & \ddots & \ddots & & & & & \\
\vdots & \ddots & t_m^u & \vdots & \ddots & & & & & & \\
 & \ddots & 0 & t_m^u & & & & & & & \\
 & & & & & & & & t_m^{u^*} & 0 & \cdots \\
 & & & & & & & & \vdots & \ddots & \ddots \\
 & & & & & & & & t_1^{u^*} & \cdots & t_m^{u^*} \\
 & & & & & t_m^u & \cdots & t_1^u & & & \\
 & & & & & 0 & \ddots & & & \mathbf{Y}_m^u & \\
 & & & & & \vdots & \ddots & t_m^u & & &
\end{array}\right]$$

where $\mathbf{X}_m^u$ and $\mathbf{Y}_m^u$ are two $m\times m$ Hermitian blocks (neither depends on n) and $t_i^u, i=0\cdots m$ are given by
$t_i^u=\left(\mathbf{H}_n^H(\mathbf{h}^u)\mathbf{H}_n(\mathbf{h}^u)\right)_{m+1+i,m+1}$
$=\sum_{k\in\{m,\ldots,0\}\bigcap\{i+m,\ldots,i\}}h_k^{u^*}h_{k-i}^u=\sum_{k=i}^{m}h_k^{u^*}h_{k-i}^u$ whose Fourier transform equals $|h^u(w)|^2$.

We introduce the $n\times n$ finite order Hermitian Toeplitz matrix

$$\mathbf{T}_n=\begin{bmatrix} t_0 & \cdots & t_m^* & 0 & \cdots \\ \vdots & \ddots & & & 0 \\ t_m & \ddots & & & t_m^* \\ 0 & & & & \\ \ddots & 0 & t_m & \cdots & t_0 \end{bmatrix}$$

where $t_k=\sum_{u=1}^{c}t_k^u$. From [4], $\sigma_n(\mathbf{T}_n)\ge min_{w\in[-\pi,\pi]}\left(\sum_{u=1}^{c}|h^u(w)|^2\right)\ge 0$ and hence $\mathbf{T}_n$ is positive definite. Furthermore, one can prove that $\mathbf{X}_m+\mathbf{Y}_m=\mathbf{T}_m$ where $\mathbf{X}_m\hat{=}\sum_{u=1}^{c}\mathbf{X}_m^u$ and $\mathbf{Y}_m\hat{=}\sum_{u=1}^{c}\mathbf{Y}_m^u$.

$$\mathbf{H}_n^H(\mathbf{h})\mathbf{H}_n(\mathbf{h})=\sum_{u=1}^{c}\mathbf{H}_n^H(\mathbf{h}^u)\mathbf{H}_n(\mathbf{h}^u)$$

$$=\mathbf{T}_{n+m}+\begin{bmatrix}\mathbf{X}_m-\mathbf{T}_m & \mathbf{0}_{m,n-m+1} & \mathbf{0}_m \\ \mathbf{0}_{n-m+1,m} & \mathbf{0}_{n-m+1} & \mathbf{0}_{n-m+1,m} \\ \mathbf{0}_m & \mathbf{0}_{m,n-m+1} & -\mathbf{X}_m\end{bmatrix} \quad (3)$$

We now prove that the matrix sequence $\mathbf{H}_n^H(\mathbf{h})\mathbf{H}_n(\mathbf{h})$ is asymptotically equivalent to the Toeplitz matrix sequence $\mathbf{T}_{n+m}$ when $n\to+\infty$, and so will be said to be asymptotically Toeplitz. First, from [4, Lemma 4.1 (4.4)], $\mathbf{T}_n$ is bounded. So is $\mathbf{H}_n^H(\mathbf{h})\mathbf{H}_n(\mathbf{h})$ as $\left\|\mathbf{H}_n^H(\mathbf{h})\mathbf{H}_n(\mathbf{h})\right\|\le\|\mathbf{T}_{n+m}\|+\left\|\begin{bmatrix}\mathbf{X}_m-\mathbf{T}_m & \mathbf{0}_m\\ \mathbf{0}_m & -\mathbf{X}_m\end{bmatrix}\right\|$. Also, from (3), we have $|\mathbf{H}_n^H(\mathbf{h})\mathbf{H}_n(\mathbf{h})-\mathbf{T}_{n+m}|$
$=\sqrt{\frac{2m}{n+m}}\left|\begin{bmatrix}\mathbf{X}_m-\mathbf{T}_m & \mathbf{0}_m\\ \mathbf{0}_m & -\mathbf{X}_m\end{bmatrix}\right|\to 0$. Hence, $\mathbf{H}_n^H(\mathbf{h})\mathbf{H}_n(\mathbf{h})$ is asymptotically equivalent to $\mathbf{T}_{n+m}$ when $n\to+\infty$.

Consequently, from Lemma 1, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n+m}\lambda_k^s(\mathbf{H}_n^H(\mathbf{h})\mathbf{H}_n(\mathbf{h}))=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n+m}\lambda_k^s(\mathbf{T}_{n+m})$$

Following the same procedure as in [4], this can be extended to the following. For all continuous functions F,

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n+m}F\left(\lambda_k(\mathbf{H}_n^H(\mathbf{h})\mathbf{H}_n(\mathbf{h}))\right)\\ =\;&\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n+m}F\left(\lambda_k(\mathbf{T}_{n+m})\right)\end{aligned}$$

Using Szegö's theorem, we prove the following

Theorem 2 *For all continuous functions F,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n+m}F\left(\sigma_k^{(n)}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}F\left(\sigma_s\sqrt{\sum_{u=1}^{c}|h^u(w)|^2}\right)\,dw$$

The following can be deduced (the proof is similar to that of [7, Corollary 3.9]) about the smallest singular value.

Theorem 3 *If $\sigma_{n+m}^{(n)}$ converges in n, then*

$$\lim_{n\to\infty}\left(\sigma_{n+m}^{(n)}\right)\le\sigma_s\min_w\left(\sqrt{\sum_{u=1}^{c}|h^u(w)|^2}\right) \quad (4)$$

4. IMPLICATIONS FOR BLIND SIMO CHANNEL IDENTIFICATION

Blind identification of a SIMO channel from its SOS is made possible when the channel is zero-coprime and its SOS are completely known ($n>m$). SOS-based blind techniques always involves the inversion of the (noise free) correlation matrix (The Linear Prediction (LP) [1] and the Outer Product Decomposition (OPD) [3] algorithms) or the determination of its kernel (the Subspace (SS) [8] algorithm). Both approaches are sensitive to the conditioning of the processed correlation matrix [11]. This conditioning is well expressed in terms of the lowest non-zero eigenvalue of the (rank deficient) correlation matrix, i.e. $\lambda_{n+m}(\mathbf{R}_n)=\left(\sigma_{n+m}^{(n)}\right)^2$. On the other hand, $\sigma_{n+m}^{(n)}$ is well approximated [5, Theorem p. 72] by $\lim_{n\to\infty}\left(\sigma_{n+m}^{(n)}\right)$. We, therefore, suggest the left-hand side of (4) as an algorithm-independent *measure* of blind identifiability.

The right-hand side of (4) is better suited to assess channel blind (un)identifiability under practical observation conditions. In fact, in cases where the right-hand side in (4) is small, the channel output covariance matrix is poorly conditioned and blind algorithms are expected to fail to identify the channel if its output is observed over a limited time duration. This bound has also the advantage of giving a spectral interpretation of channel blind identifiability.

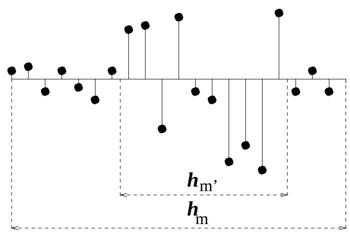


Fig. 2. Channel response with small heading and trailing terms

This bound has a further interpretation in the practical case when the channel response includes small heading and/or trailing terms (Fig. 2). The whole m-order channel response $\mathbf{h}$ can be written as the sum of an m'-order effective response $\mathbf{h}_{m'}$, $m' < m$, and a perturbation vector due to the small trailing terms [6]. If we let $h_{m'}^{u}(w)$ be the Fourier transform associated with the sub-channel $u = 1, \cdots, c$ of $\mathbf{h}_{m'}$, then $\sum_{u=1}^{c}|h^u(w)|^2 \simeq \sum_{u=1}^{c}|h_{m'}^{u}(w)|^2$ i.e., the bound in (4) is approximately the same when evaluated for $\mathbf{h}$ or $\mathbf{h}_{m'}$. When this bound is weak, it implies non-identifiability of the whole response as well as the effective response. In such case, the channel will not be identifiable whatever the assumed channel order. When assumed to be greater than m', it leads to a poorly conditioned covariance matrix because of the small trailing terms. When less than m', the identification procedure will fail because some significant terms are ignored. When equal to m', blind identification is still not possible because of the bound (and hence the conditioning of the correlation matrix) is unfavorable. Hence, while generally not tight as verified through computations, the upper bound in (4), when low, indicates *practical non identifiability* of the channel i.e., neither the channel nor a part of it can be identified from a finite observation set. Examples are given in the practical case of fractionally received bandlimited channels.

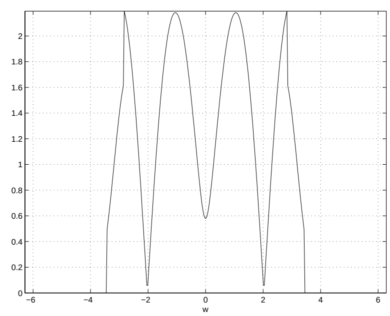
5. FRACTIONALLY SPACED BANDLIMITED CHANNELS

If the sub-channels $h^u(w), u = 1, \cdots, c$ are generated by oversampling a waveform $h(t)$ and $h(w)\hat{=}\int h(t)e^{-jwt}dt$, then $h^u(w) = \sum_l h(w-2l\pi)e^{-j(w-2l\pi)\frac{u-1}{c}}$. If $h(w)$ is bandlimited (to $[-\frac{1}{T},\frac{1}{T}]$), then $h^u(w) = h(w)e^{-jw\frac{u-1}{c}} + h(w-2\pi)e^{-j(w-2\pi)\frac{u-1}{c}}$ for $w \in [0, 2\pi]$ and it can be proved that (4) simplifies to

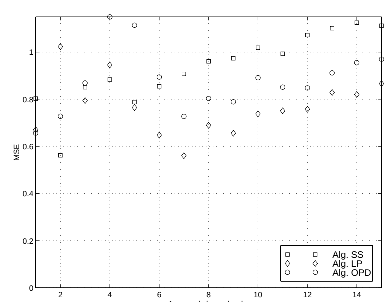
$$\lim_{n\to\infty}\left(\sigma_{n+m}^{(n)}\right) \leq \sigma_s\sqrt{c}\min_w\left(\sqrt{(|h(w)|^2 + |h(w-2\pi)|^2)}\right)$$

More commonly, $h(w)$ is a shaping filter (a raised cosine waveform most often) propagating through a frequency selective multipath channel. Some frequency components can be significantly attenuated leading the above upper bound to be weak. This justifies the poor performance of blind algorithms in identifying communication channels using fractional receivers, and concurs with remarks in [2][1]

A series of simulations was conducted with a raised cosine waveform[2] with rolloff 0.3, propagating through randomly selected multipath channels with a 4 symbol period delay spread[3]. Channels for which the upper bound of (4) was weak (≤ 0.1), such as in Fig. 3, were systematically *practically non identifiable*[4] in the sense given in Section 4. On the contrary, however, when the upper bound of (4) was not weak, no conclusion can be made. An order with which reliable identification can be performed may exist (Fig. 4) or not (Fig. 5).



(a) Frequency response $h(w)$



(b) Blind identification error

Fig. 3. The upper bound in (4) equals 0.0851.

6. CONCLUSION

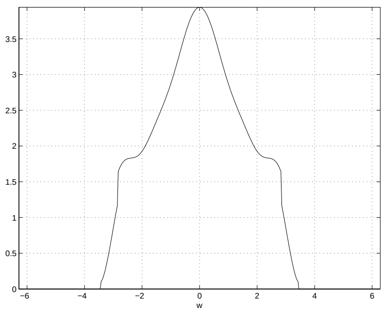
We studied the asymptotic behavior of the singular values of the SIMO channel filtering matrix when the smoothing factor tends

[1]The therein made analysis however is relative to the SS algorithm.

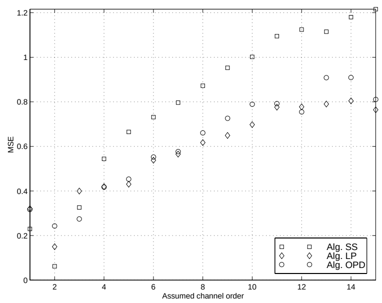
[2]The waveform response was truncated over 40 symbol periods.

[3]The direct path is not delayed and not attenuated while the number, delays and attenuations of the weaker and delayed paths are randomly chosen. The channel response was normalized so that $\|\mathbf{h}\| = 1$

[4]Identification was tried using the subspace algorithm. The channel was observed over 300 symbol periods with an SNR of $20dB$ and was $T/2$ sampled. The channel was declared non identified when the mean square error exceeds 0.1.

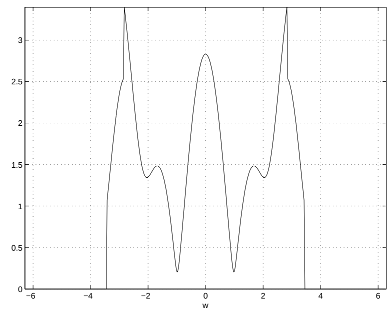


(a) Frequency response $h(w)$

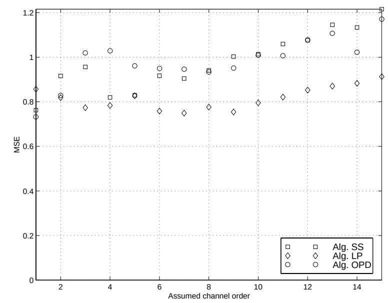


(b) Blind identification error

Fig. 4. The upper bound in (4) equals 1.0690.



(a) Frequency response $h(w)$



(b) Blind identification error

Fig. 5. The upper bound in (4) equals 0.3033.

to infinity. This is proved to be linked to the transfer functions of the sub-channels. We are interested in particular in the lowest singular value which, because it expresses the conditioning of the correlation matrix, is a relevant and algorithm-independent measure of the channel blind identifiabily from SOS. With respect to this lowest singular value, an upper bound is derived which gives a practical spectral interpretation of channel blind identifiability. The practical case of fractionally spaced bandlimited channels is studied and the proof is made that when the channel is frequency selective, it can not be blindly identified, regardless of the assumed channel order.

7. REFERENCES

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