MODELING AND EQUALIZATION OF AUDIO SYSTEMS USING KAUTZ FILTERS

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ABSTRACT

Frequency warping using allpass structures or Laguerre filters has found increasingly applications in audio signal processing due to good match with the auditory frequency resolution. Kautz filters are an extension where the frequency warping and related resolution can have more freedom. In this paper we discuss the properties of Kautz filters and how they meet typical requirements found in modeling and equalization of audio systems. Case studies include transfer function modeling of the guitar body and loudspeaker response equalization.

1. INTRODUCTION

Any bandlimited linear and time-invariant (LTI) system can be approximately modeled or inverse modeled by a digital filter. A typical case of inverse modeling in audio signal processing is equalization of response to approximate a desired target response. Any stable filter structures can be used, and although FIR and direct-form IIR structures in principle cover all LTI cases, computational precision, efficiency, controllability of response, easy design, etc., may make other filter structures more desirable.

The kernel element of simple digital filters is the unit delay, $D(z) = z^{-1}$. While it is always possible to reduce any LTI filter back to a direct-form structure, some other kernels or basis functions may be useful from theoretical or practical points of view. A generalization of the unit delay (or a cascade of unit delays) is for example an allpass filter that has flat magnitude response but frequency-dependent delay. This introduces a frequency mapping (warping) and makes it possible to look at such filter structures in a modified (warped) frequency domain.

The best known such frequency mapping is based on the bilinear conformal mapping $D_1 = (z^{-1} - \lambda)/(1 - \lambda z^{-1})$. An orthonormal formulation with such basis functions leads to *Laguerre filters* [9] which consist of a cascade of identical allpass elements preceded with a normalization filter. In many frequency-warped filter designs the normalization can be simply skipped [5].

In Laguerre type of filters there is only one degree of freedom, parameter λ, which specifies the non-uniform frequency resolution of a warped frequency domain. This is enough and useful in many cases, particularly since with careful optimization there exists a good match with the Bark scale that is used to describe the psychoacoustical frequency scale of human hearing [11]. There are many cases, however, e.g. in audio signal processing, where a more complex frequency resolution mapping is desirable.

Kautz filters [8] is an interesting extension to Laguerre filters which allow for such features. In this paper we discuss the properties of Kautz filters from the point of view of audio signal processing. After a brief theoretical background of implementation and design principles, we present two examples as case studies of using Kautz filters in modeling and inverse modeling of audio systems. In the first case we apply the method to loudspeaker response equalization. The second case deals with the modeling of guitar body impulse response. We conclude with a summary of the work and a comparison with traditional design methods.

2. THEORY OF KAUTZ FILTERS

Deducible in many ways, the lowest order rational functions, square-integrable and orthonormal on the unit circle, analytic for $|z| > 1$, are of the form [13]

$$G_i(z) = \frac{\sqrt{1 - z_i z_i^*}}{z^{-1} - z_i^*} \prod_{j=0}^{i} \frac{z^{-1} - z_j^*}{1 - z_j z^{-1}}, \quad i = 0, 1, \dots, \qquad (1)$$

defined by any set of points $\{z_i\}_{i=0}^{\infty}$ in the unit disk. Functions (1) form an orthonormal set which is complete, or a base, with a moderate restriction on the poles $\{z_i\}$ [13]. The corresponding time functions $\{g_i(n)\}_{i=0}^{\infty}$ are impulse responses or inverse z-transforms of (1). This implies that a basis representation of any causal and stable discrete-time signal or LTI system is obtained as its Fourier series expansion with respect to the time or frequency domain basis functions. These generalizations of z-transform and convolution sum representations provide linear-in-parameter models for signals and systems.

In the signal processing context, functions (1) are called Kautz functions, and they inherit their name and a specific way of deduction from a method proposed by Kautz to orthonormalize a set of continuous-time exponential components [8]. The discrete-time version can be attributed to Broome [2]. A Kautz filter is a finite weighted sum of functions (1), which clearly reduces to a transversal structure of Fig. 1. The filter structure is completely determined by a pole set $\{z_i\}_{i=0}^{N}$ and a weight vector $\mathbf{w} = [w_0 \cdots w_N]^T$. We define the filter or model order to be $N + 1$.

A Kautz filter produces real tap output signals only in the case of real poles. However, from a sequence of real or complex conjugate poles it is always possible to form real orthonormal structures. From the infinite variety of possible solutions it is sufficient to use the intuitively simple structure of Fig. 2, proposed by Broome: the second-order section outputs of Fig. 2 are *orthogonal* from which an orthogonal tap output pair if formed. Normalization terms are completely determined by the corresponding pole pair $\{z_i, z_i^*\}$ and are given by $p_i = \sqrt{(1-\rho_i)(1+\rho_i-\gamma_i)/2}$ and $q_i = \sqrt{(1-\rho_i)(1+\rho_i+\gamma_i)/2}$, where $\gamma_i = -2RE\{z_i\}$ and $\rho_i = |z_i|^2$ can be recognized as corresponding second-order

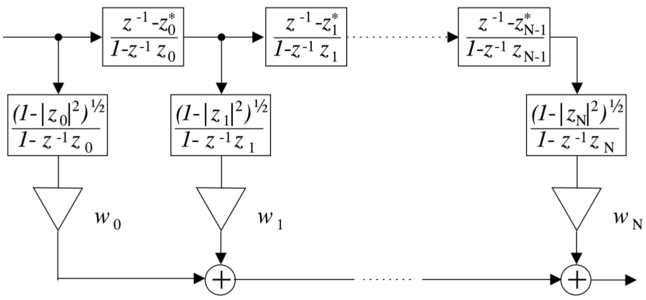

Figure 1: The Kautz filter. For $z_i = 0$ in (1) it degenerates to an FIR filter and for $z_i = a, -1 < a < 1$, it is a Laguerre filter where the tap filters are replaced by a common pre-filter.

polynomial coefficients. The construction works also for real poles but we use an obvious mixture of first- and second-order sections, if needed.

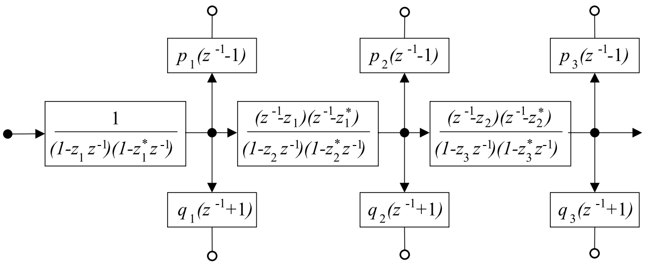

Figure 2: One realization for producing real Kautz functions from a sequence of complex conjugate pole pairs.

2.1. Modeling and inverse modeling with Kautz filters

As hinted previously, for a given system $h(n)$ or $H(z)$, Fourier coefficients provide least-square (LS) optimal parametrizations for the corresponding Kautz model, or synthesis filter, with respect to the pole set and the filter order. Evaluation of the Fourier coefficients, $c_i = (h, g_i) = (H, G_i)$, can be implemented by feeding the signal $h(-n)$ to the Kautz filter and reading the tap outputs $x_i(n) = G_i[h(-n)]$ at $n = 0$: $c_i = x_i(0)$. It should be noted that here the LS criterion is applied on the infinite time horizon and not for example in the time window defined by $h(n)$.

A straightforward interpretation for input-output-data identification of Kautz model parameters is based on *normal equations* assembled from correlation terms: if the tap outputs $x_i(n) = G_i[x(n)]$ to the input $x(n)$ are collected in the matrix $\mathbf{S}$, $[s]_{ij} = \{x_{i-1}(j-1)\}_{i,j=1}^{N,\infty}$, the weight vector is the solution of the matrix equation $\mathbf{Rw} = \mathbf{p}$, with $\mathbf{R} = \mathbf{SS}^H$ and $\mathbf{p} = \mathbf{Sy}$, where $\mathbf{y} = [y(0)\, y(1) \cdots]^T$ is the (infinite) desired output vector and H denotes complex conjugate transpose. In practice the tap output signals are truncated or windowed somehow. It should be noted that this is just the prototype LS approach, and many other types of minimization criteria may be used.

There are basically three types of interpretations for inverse modeling and equalization: (a) model the system and invert the model, (b) invert the system description and model the inverse, and (c) identify the overall system. (a) is a straightforward generalization of the moving-average model case, if we include a delay-free loop elimination method [4]. In (c) we have the impulse and non-impulse input formulations, with possible treatment for the overall delay. Here we choose (b) because then we have direct powerful means to optimize the pole positions $\{z_i\}$.

2.2. Optimization of Kautz filter design

Even in the one-pole Laguerre case it is impossible to optimize the pole position analytically. Nevertheless there are many methods that can be used in search for suitable poles $\{z_i\}$, including all-pole or pole-zero modeling, sophisticated guesses, and random or iterative search. For the structure of identical allpass blocks, a relation between optimal model parameters and error energy surface stationary points with respect to the poles may be utilized [3] as well as a classification of systems to associate systems and basis functions [12].

For a given target response, the most appealing methods could be titled as *allpass operator optimization*. As previously noted, we get the LS optimal filter weights by feeding $h(-n)$, $n = 0, \ldots, M$, to the Kautz filter and reading the tap outputs at $n = 0$. As a consequence of the orthogonality, the resulting approximation error energy $\sum_{i=N}^{\infty} |w_i|^2$ is equal to the output of the allpass operator (defined by the Kautz filter) in the finite interval $[-M, 0]$! The two methods applied in this paper can be seen as linearizations of the nonlinear optimization problem with respect to the poles, based on this energy observation [10, 1]. Apparently neither of these methods have been used in optimization of discrete-time Kautz filter structures.

3. CASE 1: LOUDSPEAKER EQUALIZATION

An ideal loudspeaker has a flat magnitude response and a constant group delay. Here we demonstrate the use of Kautz filters in pure magnitude equalization, based on an inverted target response, although direct utilization of methods (a) or (c) in Section 2.1 would produce inherent magnitude and phase equalization. The measured loudspeaker magnitude response and a derived equalizer target response are included in Fig. 3. The sample rate is 48 kHz.

Audio equalization consists typically of compensating for three different types of phenomena: slow trends in the response, sharp and local deviations, and correction of roll-offs at the band edges. This makes "blind equalization" methods ineffective. We propose that Kautz filters provide a useful alternative between "blind" and hand-tuned "parametric" equalization (with an obvious abuse of terminology).

As is well known, FIR modeling has on inherent emphasis on high frequencies on the auditorily motivated logarithmic frequency scale. Warped FIR (or Laguerre) filters release some of the resolution to the lower frequencies, providing a competitive performance with 5 to 10 times lower filter orders [6]. However, the filter order required to flatten the peaks at 1 kHz is still high, of the order 200, and in practice Laguerre models up to order 50 are able to model only slow trends in the response.

In search for lower order Kautz models we applied various methods presented in Section 2.2. AR modeling and AR modeling based ARMA modeling do not provide good pole sets. Steiglitz-McBride method (SM) of ARMA modeling produces unstable poles (for orders above 8), but for some orders omitting the unstable poles leaves a usable pole set (Fig. 3). The method proposed by McDonough and Huggins [10] produces unstable poles in the first iteration from almost any initial pole set. The method of Brandenstein and Unbehauen [1] provides stable and reasonable pole sets for orders at least up to 40. The low frequency behavior of the target response produces poles extremely close to $z = 1$, which is also the reason for troubles in the two previous methods, and some of these poles were omitted in 28th and 33th order Kautz models of Fig. 3.

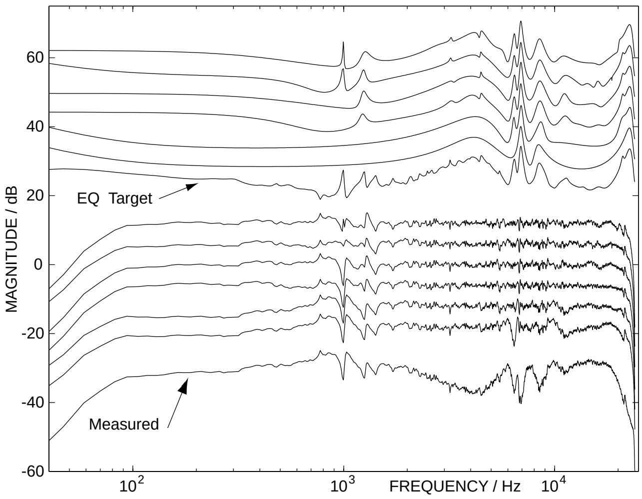

Figure 3: Displayed with offset from bottom to top: the measured impulse response, simulated equalization with Kautz models of orders 9, 15, 28, 29(SM), 33 and 34(SM), the equalizer target response, and corresponding Kautz equalizer responses.

To improve the modeling at 1 kHz, we tried to fit different pole sets to the resonances. The starting point was the 28th order Kautz filter of Fig. 3, where we omitted the one pole pair trying to model the 1 kHz region. Three pole pairs were manually tuned to the three prominent resonances, resulting in the 32th order Kautz filter of Fig. 4. To improve the modeling below the 1 kHz region we added a suitable pole pair, producing the 34th order Kautz model in Fig. 4. To lower further the filter order, we applied the same procedure to the 15th order model of Fig. 3 providing a relatively good equalization result at a much lower filter order 23 (Fig. 4).

Finally, after extensive tuning of 10 pole pairs we ended up with the 20th order model of Fig. 4. This is an ultimate approach in the sense that each (chosen) resonance is modeled with only one pole pair. Here also a compensating pole pair is placed at the low end. This is clearly one form of parametric equalization with second order blocks. However, with Kautz filters we have completely separated the choosing of the resonance structure and the (linear-in-parameter) model parametrization.

4. CASE 2: MODELING OF GUITAR BODY RESPONSE

As an example of high-order distributed-pole Kautz modeling we approximate a measured acoustic guitar body response sampled at 24 kHz (Fig. 5). The obvious disadvantage of a straightforward FIR filter implementation is that modeling of the slowly decaying lowest resonances requires very high filter orders. All-pole or pole-zero modeling are the traditional choices in improving the flexibility of the spectral representation. Perceptually motivated warped counterparts of all-pole and pole-zero modeling pay off, even in technical terms [7], but here we want to focus the modeling resolution more freely.

A 1000 tap FIR implementation was selected as a reference model, which gives a safe complexity advantage margin to a 100 to 200 order Kautz filter. Direct all-pole or pole-zero modeling were found to produce unsatisfactory pole sets $\{z_i\}$, even in searching

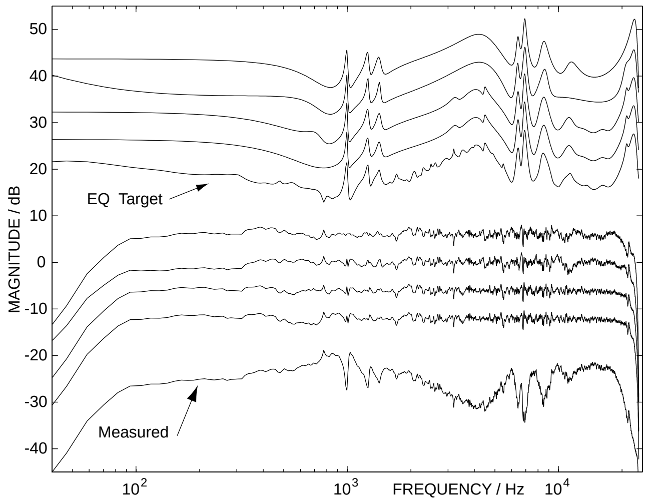

Figure 4: Equalized responses and Kautz equalizers constructed partly or purely from observing original and target responses. The Kautz filter orders are 20, 23, 34 and 32, from top to bottom, in respective block for the equalizers and the simulated equalization.

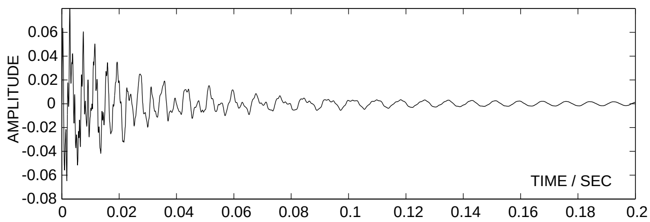

Figure 5: Measured impulse response of an acoustic guitar body.

for a low-order substructure. On the other hand, it is relatively easy to find good pole sets by direct selection of prominent resonances and proper pole radius tuning. To demonstrate the abilities of the proposed allpass optimization methods, we apply the method of Brandenstein and Unbehauen on the full filter order dimension 100 for the allpass structure. Two real poles were disregarded and the other 49 pole pairs were recognized as good representatives for the resonance structure. Based on this observation, poles were associated with different choices of prominent resonances. In Table 1 the resulting time and frequency domain normalized root-mean-square-errors (NRMS) corresponding to certain choices of block orders and number of blocks are presented, compared to the FIR 1000 case. In last two columns an *ad hoc* method to compensate the repetitive appearance of the poles is demonstrated: the pole radii are simply raised to the power of the block number.

Figure 6 presents the low frequency behavior of (16×6) and (10×10) Kautz models of Table 1, compared to the original and the 1000-tap FIR frequency response. No fine tuning is done and the poles are representatives of the original 49 pole pair set, pole pairs 1 to 8 (with increasing angles) and in the latter case omitting pairs 5 to 7. The common pole pairs correspond quite well to the five prominent resonances and the figure illustrates the tradeoff between resonance and off-resonance behavior. The Kautz filter orders are very low and an increase in orders will rapidly improve the overall spectral details, still having the emphasis on the low frequency range.

order	E_t	E_z	E'_t	E'_z
98×1	0.2140	0.1355		
50×2	0.1735	0.1200	0.1726	0.1322
24×4	0.1351	0.0951	0.1479	0.1052
16×6	0.1998	0.1689	0.1583	0.1162
10×10	0.2676	0.2102	0.1871	0.1346
FIR 1000	0.2930	0.2210		

Table 1: Time and frequency domain NRMS errors for different pole sets of Kautz filters, compared to a 1000-tap FIR, last two columns with modified pole radii.

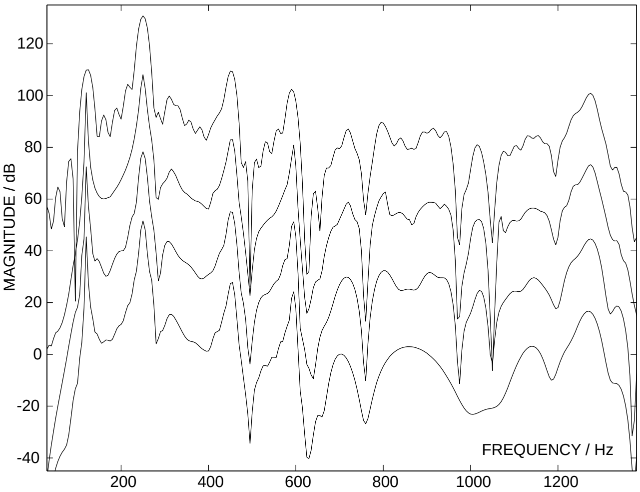

Figure 6: Displayed with offset from the top: magnitude responses up to 1380 Hz, for the 1000-tap FIR filter, the measured response, (16×6) and (10×10) order Kautz filters.

5. DISCUSSION AND CONCLUSIONS

The above cases of audio equalization and modeling were taken as challenging examples in order to show the applicability of Kautz filters. Many specific questions, such as audio relevance of the modeling details, perceptual aspects of the designs, as well as computational robustness and expense have been addresses briefly or not at all. Thus call for further investigations.

The aim of this study was to show that it is possible to achieve good modeling or equalization results with lower Kautz filter orders than with warped (Laguerre) or traditional FIR and IIR filters. In the loudspeaker equalization case Kautz filter orders of 20–30 can achieve similar results of flatness as warped IIR models of order 100–200, or much higher orders with FIR equalizers. This reduction is due to well controlled focusing of frequency resolution on both global shape and particularly on local resonant behavior. In the case of guitar body response modeling the low-frequency modes are important perceptually, and relatively low-order (about 100) Kautz filters can focus sharply on them, showing advantage over warped, IIR, and FIR designs, especially when focus is on the separate low-frequency modes of the body response.

The basic flexibility of Kautz filter designs doesn't come without complications. In this paper we have hand-tuned the pole set $\{z_i\}$ of Eq. 1 to yield superior modeling with low orders. There are numerous possible techniques and strategies to search for an optimal model for a given problem, and different tasks may be solved best with different approaches. The cases investigated here just hint general guidelines, and fully automated search for optimal solution even in the present cases requires further work. However, we have demonstrated the potential applicability of Kautz filters. They are found flexible generalizations of FIR and Laguerre filters, providing IIR-like spectral modeling capabilities with well-known favorable properties resulting from the orthonormality. The competitiveness compared to Laguerre modeling is based on the fact that the generalization step imposes little or no extra computation load at runtime, even if the design phase may become more complicated.

MATLAB scripts and demos related to Kautz filter design can be found at `www.acoustics.hut.fi/software/kautz`.

6. ACKNOWLEDGEMENTS

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