SPREAD OFDM PERFORMANCE WITH MMSE EQUALIZATION

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ABSTRACT

In this paper, upper bounds on the bit error probability of an OFDM/CDMA system in a single user context are derived for a linear minimum mean-square-error (MMSE) receiver structure. All the calculus are performed in presence of a frequency-selective Rayleigh fading channel both in a coded (convolutive Forward Error Correcting scheme) and uncoded scenario. Surprisingly, we show that coded SOFDM outperforms COFDM if the signal to noise ratio is greater than a given threshold which **only** depends on **code rate** and not on the code structure. These theoretical results provide tools to optimally combine coding and spreading for increasing the spectral efficiency of OFDM/CDMA schemes.

1. INTRODUCTION

A multi-carrier OFDM system [1] using a Cyclic Prefix for preventing inter-block interference is known to be equivalent to multiple flat fading parallel transmission channels in the frequency domain. In such a system, the information sent on some carriers might be subject to strong attenuations and could be unrecoverable at the receiver. This has motivated the proposal of more robust transmission schemes combining the advantages of CDMA with the strength of OFDM known as OFDM-CDMA [2], in which the information is spread across all the carriers by a pre-coding unitary matrix (e.g. the Walsh-Hadamard). This combination increases the overall frequency diversity of the modulator, so that unreliable carriers can still be recovered by taking advantage of the subbands enjoying a high Signal to Noise Ratio (SNR). Although originally proposed for a multiuser access scheme, this concept is extended to all single user OFDM systems and is referred in the sequel as Spread OFDM (SOFDM). Due to the inter-carrier interference generated by the spreading, the frequency domain channel transfer function of a single antenna SOFDM system can be modeled using a full MIMO flat fading (scalar) matrix. Even though optimal maximum-likelihood (ML) detector clearly outperforms MMSE receivers, for complexity and practical implementation issues, this paper will only focus on MMSE equalizers. [3][4] showed *by simulations* that the diversity improvement induced by the spreading matrix is considerably reduced when coding is applied. Indeed, the high redundancy of low code rates (which already performs a kind of spreading) in conjunction with the suboptimality of the MMSE receiver reduces the performance with respect to COFDM. The purpose of this paper is to analyze more rigorously this observation.

Therefore, analytic expressions of upper bounds of the bit error probability provided by the MMSE receiver are derived for a Rayleigh fading channel, both in the uncoded and coded scenario, when the number N of carriers is large enough. Based on these results, we study the influence of the coding rate and determine whether the MMSE receiver still outperforms COFDM when combined with coding and interleaving. It is shown in particular that the performance of COFDM with respect to SOFDM depends **only** on the **code rate**. More precisely, for any code of rate R, coded SOFDM outperforms COFDM if E_b/N_0 is greater than a given threshold *which only depends on* R and not on the code structure. This result is believed to be quite useful since it provides new insights for choosing between a COFDM or coded SOFDM modulation scheme in a given context.

The system and channel model are described in section 2, followed by a theoretical analysis of the MMSE receiver in the uncoded case (section 3) and in the coded scenario (section 4). Finally, section 5 is devoted to simulations which confirm our theoretical claims. Some conclusions are provided in Section 6.

2. SOFDM TRANSCEIVER MODEL

In the following, upper (lower boldface) symbols will be used for matrices (column vectors) whereas lower symbols will represent scalar values, $(.)^T$ will denote transpose operator, $(.)^\star$ conjugation and $(.)^H = \left((.)^T\right)^\star$ hermitian transpose. $I_{P\times P}$ will represent the $P \times P$ identity matrix.

Overall system model: Since a N carrier OFDM system [1] using a cyclic prefix is equivalent in the frequency domain to N flat fading parallel transmission channels, the baseband discrete-time block equivalent model of a SOFDM system can be depicted in figure 1. Actually the $N\times 1$ received vector at time n, $\mathbf{r}(n) = (r_1(n), \cdots, r_N(n))^t$ can be expressed as a function of the emitted symbol vector at time n, $\mathbf{s}(n) = (s_1(n), \cdots, s_N(n))^t$ and of the additive noise $\mathbf{w}(n) = (w_1(n), \cdots, w_N(n))^T$ vector using a MIMO flat fading channel matrix $M(n)$:

$$\mathbf{r}(n) = M(n)\mathbf{s}(n) + \mathbf{w}(n) \tag{1}$$

where $M(n)$ consists in the product of the spreading matrix T (usually a Walsh Hadamard transform), which can be interpreted as a source of inter-carrier interference, by the diagonal matrix $C(n) = \mathrm{diag}(c_1(n), \cdots, c_N(n))$ of the frequency domain channel attenuations at time n: $M(n) = C(n)T = (\mathbf{m}_1(n), \cdots, \mathbf{m}_N(n))$. In the following, the QPSK symbols $s_i(n)$ are assumed to be independent and identically distributed (i.i.d.). We will also assume channel knowledge and perfect channel synchronization at the receiver.

3. THEORETICAL PERFORMANCE OF UNCODED SOFDM WITH MMSE EQUALIZATION

First let study the performance of OFDM and SOFDM with MMSE equalization in the uncoded case. These results will be used to obtain upper bounds on the bit error probability in the coded case. In the following, we will assume that the channel coefficients $(c_i(n))_{i=1,\dots,N,n\in\mathbb{Z}}$ are independent identically distributed circular complex Gaussian random variables with variance δ^2. The independence assumption can be justified when assuming that for COFDM (time and frequency) interleaving is performed such that the emitted symbols are transmitted on distinct frequency carriers, separated by more than the frequency coherence channel bandwidth. In the case of SOFDM, the spread symbols $x(k)$ are time and frequency interleaved on independant carriers (fig.1). However, in this case, a decoding delay incurs at the receiver to retrieve the N symbols x(k) and perform block equalization. In this section, since dependence on time is not useful, we will omit the time index for sake of simplicity. Denote by $(a_i)_{i=1,\dots,N}$ the sequence defined by $a_i = |c_i|$. This is an i.i.d sequence of Rayleigh distributed random variables.

3.1. Uncoded OFDM

Let first recall the performance of uncoded OFDM. The expression of the received signal on the k^{th} carrier is $r_k = c_k s_k + w_k$ where the additive noise sequence $(w_k)_{k=1,\dots,N}$ is white, and Gaussian circular with variance σ^2. Since QPSK symbols are transmitted, the conditional bit error probability given c_k is:

$$P_{\text{OFDM}/c_k} = Q\left(\sqrt{\frac{a_k^2}{\sigma^2}}\right)$$

By defining $\lambda = \frac{\delta^2}{\sigma^2}$, the average error probability is given by:

$$\begin{aligned}\overline{P_{\text{OFDM}}} &= \int_0^\infty Q\left(\sqrt{\frac{a_i^2}{\sigma^2}}\right)\frac{2a_i}{\delta^2}e^{-\frac{a_i^2}{\delta^2}}\,da_i \\ &= \frac{1}{2}[1-\sqrt{\frac{\lambda}{\lambda+2}}]\end{aligned}$$

3.2. SOFDM with Walsh Hadamard spreading matrix

A SOFDM transmission scheme with a Walsh Hadamard spreading matrix and Wiener Filtering at the receiver is considered here. We will focus on its asymptotic performance in presence of a large number of carriers. The output of the Wiener filter is the vector $\mathbf{y}$ given by $\mathbf{y} = G\mathbf{r}$, where the matrix G is defined as

$$\begin{aligned}G &= \text{argmin}_W \|W^H\mathbf{r}-\mathbf{s}\|^2 \\ &= T^H\text{diag}\left(\frac{c_1^*}{|c_1|^2+\sigma^2},\dots,\frac{c_N^*}{|c_N|^2+\sigma^2}\right)\end{aligned}$$

Such a decoding strategy is motivated by its very simple implementation (scalar channel equalization followed by a Walsh Hadamard ($T^H = T$) matrix multiplication). The Wiener estimate y_k of symbol s_k is easily seen to be equal to $y_k = as_k + \alpha_k$ where:

$$a = \frac{1}{N}\sum_{i=1}^N \frac{a_i^2}{a_i^2+\sigma^2}$$

$$\alpha_k = \sum_{i=1}^N \frac{c_i^* T_{i,k}^*}{(a_i^2+\sigma^2)}w_i + \sum_{j=1 j\neq k}^N \left(\sum_{l=1}^N \mu_l T_{l,k}^* T_{l,j}\right)s_j$$

$T_{l,k}$ is the (l,k) term of the Walsh Hadamard matrix, ie $T_{l,k} = \pm\frac{1}{\sqrt{N}}$ and $\mu_l = \frac{a_l^2}{a_l^2+\sigma^2}$. It is shown in [5] that the distribution of the residual interference-plus noise (SINR) at the output of a linear MMSE filter can be considered as Gaussian if N is large enough. After some straightforward calculations, we get that its variance Δ^2 is given by

$$\Delta^2 = \frac{\sigma^2}{N}\sum_{i=1}^N \frac{a_i^2}{(a_i^2+\sigma^2)^2} + \sum_{j=1 j\neq k}^N |\sum_{l=1}^N \mu_l T_{l,k}^* T_{l,j}|^2 \tag{2}$$

The first term of the righthandside of (2) represents the variance of the contribution of the additive noise to the Wiener estimate, and its second term the variance of the multiuser interference generated by the Wiener filter. Since $\sum_{j\neq k} T_{l1,j}T_{l2,j}^* = \delta(l1-l2) - T_{l1,k}T_{l2,k}^*$, this last term can be rewritten as:

$$\text{MUI} = \frac{1}{N}\sum_{l=1}^N \mu_l^2 - \left(\frac{1}{N}\sum_{l=1}^N \mu_l\right)^2$$

We now evaluate the behavior of a and Δ^2 when $N \to +\infty$ in order to obtain interpretable expressions. a coincides with the empirical mean $\frac{1}{N}\sum_{l=1}^N \mu_l$. Therefore, it converges almost surely toward the mathematical expectation of variables $(\mu_l)_{l=1,\dots,N}$. Each random variable a_l^2 is χ^2 distributed with two degrees of freedom. Therefore,

$$E(\mu_l) = \int_0^\infty \frac{x}{(x+\sigma^2)}\frac{1}{\delta^2}e^{-\frac{x}{\delta^2}}\,dx.$$

In the following, denote by $E_i(x) = \int_x^\infty \frac{e^{-u}}{u}du$ the so-called exponential integral function, and $F(x) = 1 - \frac{1}{x}e^{\frac{1}{x}}E_i(\frac{1}{x})$. It is easily seen that $E(\mu_l)$ coincides with $F(\lambda)$ where we recall that $\lambda = \frac{\delta^2}{\sigma^2}$ represents the (average) signal to noise ratio. The variance Δ^2 of $w(k)$ has a similar behavior when $N \to +\infty$, and it is possible to show that

$$\lim_{N\to+\infty} \Delta^2 = F(\lambda)(1-F(\lambda)) \text{ almost surely} \tag{3}$$

These calculations allow the explanation of the good performance of SOFDM over OFDM. In OFDM, the amplitude carrying symbol s_k is a random variable with a Rayleigh distribution while in SOFDM after MMSE equalization, symbol s_k is carried by the term a which can be considered as deterministic when N is chosen large enough. SOFDM after MMSE equalization thus transforms the N parallel fading channels of OFDM into N equivalent Gaussian channels with signal to noise ratio $\Delta_1 = \frac{\Delta^2}{a^2}$. We note that similar conclusions are drawn in [6] for apparently a different (but actually formally equivalent) context of multidimensional QAM constellations systems. However, the analysis of [6] only focused on the ML detector behavior of the symbol sequence $(s_k)_{k=1,\dots,N}$. Our calculus show that, at least when N is large enough, the bit error probability of uncoded SOFDM after MMSE equalization is given by

$$\boxed{\overline{P_{\text{SOFDM}}} = Q\left(\sqrt{\frac{a^2}{\Delta^2}}\right) = Q\left(\sqrt{\frac{F(\lambda)}{(1-F(\lambda))}}\right)} \tag{4}$$

The results provided in this paper for QPSK symbols can be of course extended to all other types of constellations.

4. THEORETICAL PERFORMANCE OF COFDM AND CODED SOFDM WITH MMSE EQUALIZATION

In classical standardized OFDM systems, the incoming input bitstream is first convolutionally encoded with a code rate of R, interleaved and punctured. The resulting bits are then mapped onto a QPSK constellation for forming symbols that are distributed over all the carriers. Unfortunately, in SOFDM the convolutional encoder cannot be applied prior to the carrier symbol allocation without resulting in an extremely complex Viterbi decoding. Otherwise the metrics calculation could not be processed on a per carrier basis due to the inter-carrier noise correlations introduced by the despreading of the received samples and this would exponentially increase the number of states of the Viterbi Algorithm trellis. This is why for SOFDM, the same coding is applied on each of the carriers independently.

4.1. COFDM

It is assumed that a soft-output Viterbi algorithm is used to decode the transmitted bits. In order to specify on which signal the Viterbi algorithm operates, we denote $(r(m))_{m\in\mathbb{Z}}$ the sequence obtained after parallel to serial conversion of the received sequence after symbol de-interleaving, i.e. the scalar sequence $\ldots r_1(n), r_2(n), \ldots, r_N(n), r_1(n+1), r_2(n+1), \ldots, r_N(n+1), \ldots$. Same notation is applied to variables c, s and w so that we obtain: $r(m) = c(m)s(m) + w(m)$. In COFDM systems equalization is performed by applying the real and imaginary part operators to sequence $(\sqrt{2}\frac{c(m)^*}{|c(m)|} r(m))_{m\in\mathbb{Z}}$, the corresponding real sequence being finally de-interleaved. Thanks to the de-interleaver, the resulting real valued discrete-time signal is denoted by $z(m) = \alpha(m)b(m) + u(m)$ where α represents an i.i.d sequence of Rayleigh distributed random variables of second order moment δ^2, b is the BPSK sequence obtained by mapping the bits generated by the convolutional encoder, and u is an i.i.d. sequence of real Gaussian random variables of variance σ^2. The Viterbi decoding algorithm processes sequence z. In order to evaluate upper bounds on the BER, one usually first evaluates the probability $\overline{P_{COFDM}}(d)$ of deciding P_1 instead of P_0 where P_0 is the path of the Viterbi algorithm trellis associated to the transmitted sequence and P_1 is a path which differs by d bits from P_0. Following [7], the probability of that event, is given by

$$\overline{P_{\mathrm{COFDM}}}(d) = E\left(Q\left(\sqrt{\frac{\sum_{m=1}^{d}\alpha_m^2}{\sigma^2}}\right)\right)$$

Using that $\sum_{m=1}^{d}\alpha_m^2$ is χ^2-distributed ($2d$ degrees of freedom), we get that

$$\overline{P_{\mathrm{COFDM}}}(d) = \int_0^\infty Q\left(\sqrt{\frac{y}{\sigma^2}}\right)\frac{1}{\delta^{2d}(d-1)!}y^{d-1}e^{-\frac{y}{\delta^2}}\,dy$$

Define the code rate as R and $\frac{Eb}{No}$ as $\lambda_b = \frac{\lambda}{2R}$ ($\lambda = \frac{\delta^2}{\sigma^2}$). After some calculus (see [7]), the probability $\overline{P_{\mathrm{COFDM}}}(d)$ can be written as

$$\begin{aligned}\overline{P_{\mathrm{COFDM}}}(d) &= \frac{1}{2}[1-\sqrt{\frac{R\lambda_b}{R\lambda_b+1}}] \\ &- \sum_{l=1}^{d-1} C_{2l-1}^{l}\sqrt{\frac{R\lambda_b}{R\lambda_b+1}}\frac{1}{(4(R\lambda_b+1))^l}\end{aligned}$$

The evaluation of the probability of the above error events leads to the following BER upper bound:

$$\boxed{\overline{P_{\mathrm{COFDM}}} \leq \sum_{d=dmin}^{\infty}\frac{\beta_d}{L}\overline{P_{\mathrm{COFDM}}}(d)} \quad (5)$$

Here, d_{min} is the minimal distance of the code, L is the number of input bits in the encoder and β_d is the number of incorrectly decoded information bits, for each possible incorrect path differing from the correct one by d bits.

4.2. Coded SOFDM with Walsh Hadamard Spreading

In the context of coded SOFDM, one Viterbi decoder is applied on each subband k and processes the real and imaginary parts of the de-interleaved signal $z_k(m)$, output of the Wiener filter. According to section 3.2, $z_k(m) = ab_k(m) + u_k(m)$ where a is the positive amplitude evaluated in section 3.2, b_k is the BPSK sequence obtained by mapping the bits at the output of the convolutional encoder, and u_k is a real Gaussian i.i.d. of variance σ^2. Since a is deterministic, the bit error probability in the pairwise comparison of two paths that differ by d bits is given by :

$$\begin{aligned}\overline{P_{\mathrm{CSOFDM}}}(d) &= P\left(W(0,1) > \frac{\sum_{l=1}^{d}a^2}{\sqrt{\sum_{l=1}^{d}a^2\Delta^2}}\right) \\ &= Q\left(\sqrt{\frac{dF(2R\lambda_b)}{(1-F(2R\lambda_b))}}\right)\end{aligned}$$

$W(0,1)$ is zero mean Gaussian variable with unit variance. The overall error probability is thus bounded by:

$$\boxed{\overline{P_{\mathrm{CSOFM}}} \leq \sum_{d=dmin}^{\infty}\frac{\beta_d}{L}\overline{P_{\mathrm{CSOFDM}}}(d)} \quad (6)$$

4.3. COFDM versus coded SOFDM.

The most important point of this paper lies in the observation that for any rate R convolutional code, the difference $l(d) = \overline{P_{\mathrm{COFDM}}}(d) - \overline{P_{\mathrm{CSOFDM}}}(d)$ is positive for all $d > 2$ if the signal to noise ratio λ_b is greater than a threshold, which **only** depends on R and seems independant of d. This claim is illustrated in figures 3 and 4 in which $l(d)$ is plotted versus the signal to noise ratio for several values of $d = 3, \ldots, 10$. Figures 3 and 4 correspond to rates of $3/4$ and $1/2$ respectively and the corresponding thresholds are about $7.5dB$ and $5dB$. We have also calculated $l(d)$ for several other rates (not presented here), and the conclusions drawn for $R = 3/4$ and $R = 1/2$ are still valid. Although we were not yet able to prove analytically our claim related to the behavior of $l(d)$, all the experiments allow us to conjecture that this statement is valid. Formulas (5) and (6) in conjunction with the above properties of $l(d)$ imply that for any code of rate R, coded SOFDM outperforms COFDM if the signal to noise ratio is higher than a threshold which depend only on R, and not of the particular chosen code. Our analysis thus confirms the experimental results observed by simulation in [3].

5. SIMULATION RESULTS

In this section, we sustain our theoretical claims by numerical simulations. In all the following experiments, the number N of carriers is equal to 64, and the coefficients $(c_k)_{k=1,\ldots,N}$ are independent.

Uncoded Case: We first show that the formulas of section 3.2 are in agreement with the numerical results. In figure 2, we compare the BER given by formula 3.2 with the BER evaluated by a Monte Carlo simulations for BPSK (the performance is derived according to section3.2) and QPSK constellations. The theoretical curve as as well as the real context curve match pretty well.

Coded case: Figure 5 plots the simulated performance of two rate $1/2$ and $3/4$, constraint length $K = 7$ convolutional encoders. It confirms the theoretical behavior inferred from figures 3 and 4: for each code, a threshold is observed which appears identical to the theoretical predictions of figures 3 and 4.

6. CONCLUSION

In this paper, theoretical tools for selecting between a COFDM or coded SOFDM modulation scheme in a given context are provided. We have derived efficient bounds to evaluate the bit-error probability knowing the transfer function of the code for coded OFDM and SOFDM schemes. It has been shown that coded SOFDM outperforms COFDM if the signal to noise ratio is greater than a threshold which only depends on R, and not on the code structure.

7. REFERENCES

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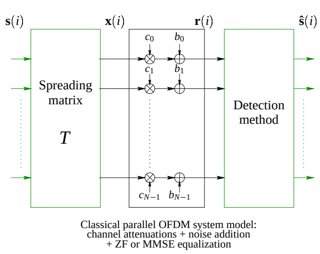

Fig. 1. Classical frequency domain SOFDM transmitter

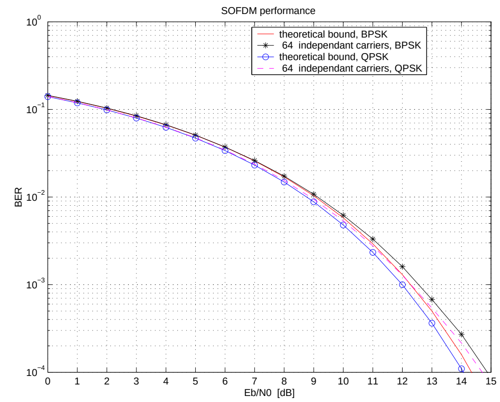

Fig. 2. SOFDM performance

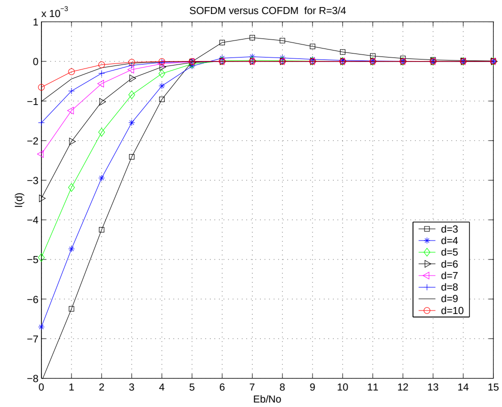

Fig. 3. rate 3/4

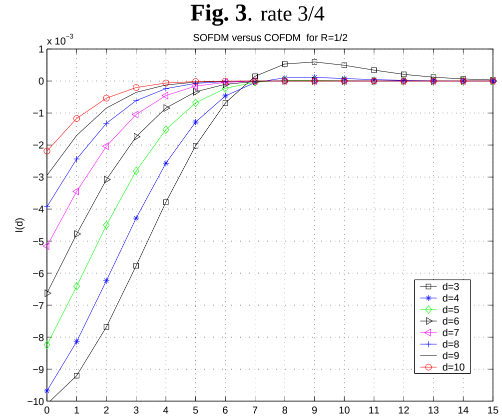

Fig. 4. rate 1/2

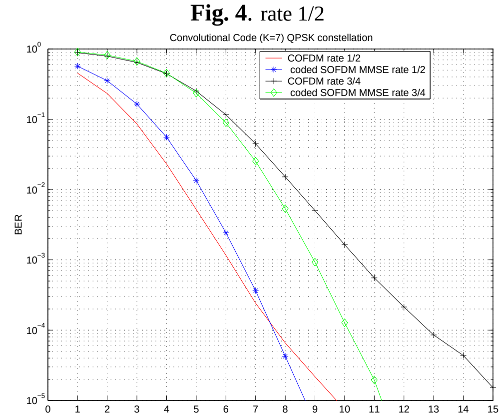

Fig. 5. Coded SOFDM-COFDM