PHYSICAL MODELING OF DRUMS BY TRANSFER FUNCTION METHODS

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ABSTRACT

Multidimensional (MD) physical systems are usually given in terms of partial differential equations (PDEs). Similar to one-dimensional systems, they can also be described by transfer function models (TFMs). In addition to including initial and boundary conditions as well as excitation functions exactly, the TFM can also be discretized in a simple way. This leads to suitable implementations for digital signal processors. Therefore it is possible to implement physics based digital sound synthesis algorithms derived from TFMs in real-time. This paper extends the recently presented solution for vibrating strings with one spatial dimension to two-dimensional drum models.

1. INTRODUCTION

Digital sound synthesis based on physical modeling has gained significant interest in the last two decades. By modeling the sound production mechanisms, traditional instruments can be understood and reproduced more intuitively than with classical signal based methods.

To keep the physical model of the sound production mechanism practicable only the dominating effects and the main vibrating structures, like the drumhead of a drum, are considered. All other surrounding material and insignificant physical behavior are neglected or are treated by fixed one-dimensional transfer functions. The analysis of this main vibrating structure using basic physical laws leads to MD models in form of PDEs. These PDEs include temporal and spatial derivatives, initial and boundary conditions and in most cases excitation functions.

A common way to solve PDEs is the finite difference method [1]. It replaces differential operators by difference functions. To keep this approximation close to the exact solution, the step sizes have to be small. This results in computationally inefficient algorithms, especially for models with two or more spatial dimensions.

A more effective way is the digital waveguide method. It simplifies the PDE to the wave equation that has a closed solution in terms of a forward and a backward traveling wave. These waves can be realized by delay lines. The effects of the other terms of the PDE are approximated by including digital filters of low order into the delay lines [2]. On the one hand this is computationally very effective for the one-dimensional case. But on the other hand there are several disadvantages, for example the loss of the physical meaning of the parameters. For systems with two or more space coordinates the waveguide method is equivalent to the finite-difference method [3] with similar disadvantages given above.

This paper demonstrates the recently presented method of using TFMs for the solution and simulation of MD physical phenomena. We extend the one-dimensional problem of a vibrating string, already solved in [4], to a two-dimensional drumhead vibration. We consider transversal waves with dispersion, frequency independent and frequency dependent losses in the drumhead.

This paper is organized as follows: Section 2 describes the physical model of the drum, section 3 gives a short description of the functional transformation method in general. Section 4 applies this method to a lossy and dispersive drumhead and section 5 presents simulation results.

2. PHYSICAL MODELS OF DRUMS

In this chapter we present the physical model of a vibrating membrane in form of a PDE. For the solution of this PDE we also need boundary and initial conditions. To keep the example simple, we assume a linear behavior and a homogeneous material.

2.1. The Partial Differential Equation

Assuming a two-dimensional membrane, the well known wave-equation is the main part of the PDE. All other effects and the excitation are added to the wave-equation and can be derived from the basic laws of elasticity. The PDE describing the vibration of a membrane with dispersion, losses and an excitation function can be written as follows [5]:

$$S^4\nabla^4 z - c^2\nabla^2 z + \frac{\partial^2}{\partial t^2}z + d_1\frac{\partial}{\partial t}z + d_3\frac{\partial}{\partial t}\nabla^2 z = v \tag{1}$$

$z = z(x_1, x_2, t)$ is the time- and space- dependent deflection and $v = v(x_1, x_2, t)$ describes the excitation-function. The spatial coordinate system (x_1, x_2) is chosen according to the shape of the drum. ∇ denotes derivative with respect to these space coordinates.

The first term of (1) containing the material dependent constant S^4 denotes dispersion resulting from the bending resistibility of the membrane. The next two terms of (1), the second order spatial and second order temporal derivatives, constitute the 2D wave equation. The constant c^2 is the material dependent speed of sound. d_1 is a frequency independent and d_3 a frequency dependent decay variable. They are derived from a simplified model [1] that includes not only the losses in the air but also those inside the membrane and those due to the coupling to the resonance body.

2.2. Boundary conditions

The shape of the membrane is defined by its boundary, a closed curve B around an area A. The number of boundary conditions is determined by the highest order of spatial derivatives of the PDE. In case of fourth order spatial derivatives, two boundary conditions must be given. If we consider a fixed membrane at the boundary, the deflection (2) and the skewness (3) are zero on B [5]. Hence the two boundary conditions are:

$$z(x_1, x_2, t)\,\big|_{(x_1,x_2)\in B} = 0 \tag{2}$$
$$\nabla^2 z(x_1, x_2, t)\,\big|_{(x_1,x_2)\in B} = 0 \tag{3}$$

2.3. Initial conditions

Since (1) has second order time derivatives, two initial conditions are required. We consider here that the deflection (4) and its first derivative (5) are given at the beginning. To model the excitation of the membrane by an initial velocity v_0 at one point (x_{10}, x_{20}), we set the deflection (4) to zero and the velocity (5) to a spatial impulse function γ_0.

$$z(x_1, x_2)\,|_{t=0} = 0 \tag{4}$$
$$\frac{\partial}{\partial t}z(x_1, x_2)\,|_{t=0} = v_0\gamma_0(x_1 - x_{10}, x_2 - x_{20}) \tag{5}$$

2.4. Concise notation

To simplify the description of the general procedure of the functional transformation method, a concise notation is introduced [4, 6]. The PDE (1) can be written as follows:

$$D\{z(x_1, x_2, t)\} + L\{z(x_1, x_2, t)\} + \\ + W\{z(x_1, x_2, t)\} = v(x_1, x_2, t) \tag{6}$$

where the operator D includes all temporal derivatives, the operator L all spatial derivatives and the operator W the mixed derivatives.

The two boundary conditions (2) and (3) as well as the initial conditions (4) and (5) can be written in vector form:

$$\mathbf{f}_b\{z\} = \left[\begin{array}{c} z \\ \nabla^2 z \end{array}\right]\Bigg|_{(x_1,x_2)\in B} = \mathbf{0} \tag{7}$$

$$\mathbf{f}_i\{z\} = \left[\begin{array}{c} z \\ \frac{\partial}{\partial t}z \end{array}\right] = \left[\begin{array}{c} z_{i0}(x_1, x_2) \\ z_{i1}(x_1, x_2) \end{array}\right] = \mathbf{z}_i(x_1, x_2) \tag{8}$$

3. FUNCTIONAL TRANSFORMATION METHOD

The basic idea of the functional transformation method is well known from time-dependent ordinary differential equations (ODEs). The Laplace transformation is used to eliminate the time derivatives and to include the initial conditions. For PDEs the Sturm-Liouville (SL) transformation is used for the space variables. It transforms the resulting boundary value problem into the spatial frequency domain. Solving this algebraic equation for the output variable, we obtain a MD TFM. After inverse transformations and discretization of the TFM the resulting system is suitable for computer simulation.

3.1. Transformation with respect to time

By applying the Laplace transformation to (6-8) we obtain a boundary value problem with spatial derivatives only. According to (1) the vibrating membrane can be described with

$$s^2 Z(x_1, x_2, s) - s z_{i0}(x_1, x_2) - z_{i1}(x_1, x_2) \\ + L\{Z(x_1, x_2, s)\} + sW_L\{Z(x_1, x_2, s)\} \\ - W_L\{z_{i0}(x_1, x_2)\} = V(x_1, x_2, s)\,, \tag{9}$$

$$\mathbf{f}_b\{Z(x_1, x_2, s)\} = \mathbf{0}\,, \tag{10}$$

where the operator $W\{z\}$ is split into two operators $W\{z\} = W_D\{W_L\{z\}\}$ with only temporal derivatives in W_D and only spatial derivatives in W_L.

3.2. Transformation with respect to space

The SL transformation is defined similar to the Laplace transformation but with finite integration limits and therefore it uses a different and a priori unknown transformation kernel $K(\beta, x_1, x_2)$ with the spatial frequency variable β.

$$\begin{aligned}\bar{Z}(\beta, s) &= \mathcal{T}\{Z(x_1, x_2, s)\} \\ &= \int_A Z(x_1, x_2, s) K(\beta, x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \qquad (11)\end{aligned}$$

To obtain an algebraic equation after SL transformation of (9) the transformation kernel $K(\beta, x_1, x_2)$ must satisfy the so called Sturm-Liouville problem [6]:

$$L\{K\} + sW_L\{K\} = \beta^4 K\,, \qquad (12)$$

$$\mathbf{f}_b\{K\} = \mathbf{0}\,. \qquad (13)$$

Its solution depends on the shape of A and cannot be solved analytically in most cases. But solving (12,13) with numerical methods leads to a transformation kernel K with which the spatial transformation (11) is defined. Applying this transformation to (9), we obtain an algebraic equation that can be solved for the output variable $\bar{Z}$.

$$\bar{Z}(\beta, s) = \frac{z_{i1}(\beta) + \bar{V}(\beta, s)}{s^2 + d_1 s + \beta_\mu^4} \qquad (14)$$

$$\bar{z}_{i1}(\beta) = v_0 K(\beta, x_{10}, x_{20}) \qquad (15)$$

$$\bar{V}(\beta, s) = \iint_A V(x_1, x_2, s) K(\beta, x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \qquad (16)$$

3.3. Inverse transformation

To apply the inverse SL transformation a short look at generalized Fourier series is needed. According to [7] every piecewise continuous function $z(x_1, x_2)$ in the bounded area A can be described as a generalized Fourier series

$$z(x_1, x_2) = \sum_\mu c_\mu K_\mu(x_1, x_2) \qquad (17)$$

with orthogonal functions $K_\mu(x_1, x_2)$. The coefficients c_μ can be calculated with

$$c_\mu = \int_A z(x_1, x_2) K_\mu(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2\,. \qquad (18)$$

Since SL problems possess orthogonal eigenfunctions K and discrete eigenvalues β_μ, $\mu \in \mathbb{N}$, we can see that the SL transformation is a generalized Fourier series. The forward transformation corresponds to (18) with $c_\mu = \bar{z}(\beta_\mu)$ and the inverse transformation is performed by (17).

Thus, inverse Laplace transformation and inverse SL transformation yield the solution of (1) in the form of an orthogonal series

$$z(x_1, x_2, t) = \sum_\mu \frac{\bar{z}(\beta_\mu, t) K(\beta_\mu, x_1, x_2)}{\|K(\beta_\mu, x_1, x_2)\|_2^2}\,. \qquad (19)$$

The inverse SL transformation (19) is computed numerically by a parallel arrangement of recursive discrete systems. They are realized efficiently on digital signal processors.

4. DRUM MODELS

Due to the dependence of $K(\beta_\mu, x_1, x_2)$ on the shape of the membrane, the SL transformation has to be calculated separately for different shapes. This is shown for a rectangular and for a circular membrane.

4.1. Rectangular membrane

The rectangular membrane is the easiest shape for the spatial transformation. Cartesian coordinates (x, y) can be chosen for (x_1, x_2) and the SL problem can be solved with a separation of the spatial variables.

$$K(\beta_\mu, x, y) = K_x(\beta_\mu, x) \cdot K_y(\beta_\mu, y) \qquad (20)$$

This approach reduces the two-dimensional problem to two one-dimensional problems, that are very similar to the string models presented in [4]. The transformation kernel and the discrete spatial frequency variables $n_x, n_y \in \mathbb{N}$ result in

$$K(n_x, n_y, x, y) = \sin\left(\frac{n_x \pi}{l_x}\right) \sin\left(\frac{n_y \pi}{l_y}\right), \qquad (21)$$

$$\begin{aligned}\beta_\mu^4(n_x, n_y) &= -\left(d_3 s - c^2\right)\left(\frac{n_x^2\pi^2}{l_x^2} + \frac{n_y^2\pi^2}{l_y^2}\right) + \\ &\quad + S^4\left(\frac{n_x^4\pi^4}{l_x^4} + \frac{n_y^4\pi^4}{l_y^4}\right) \qquad (22)\end{aligned}$$

4.2. Circular membrane

Much more interesting are circular membranes. They are a good approach for real drums and are very different from the string model.

Here polar coordinates with radius r and angle φ are chosen. The boundary of the membrane is given by the radius R. The resulting SL problem is

$$S^4 \nabla^4 K - c^2 \nabla^2 K + d_3 s \nabla^2 K = \beta_\mu^4 K\,, \qquad (23)$$

$$K(\beta_\mu, r, \varphi)\,|_{r=R} = 0\,, \qquad (24)$$

$$\nabla^2 K(\beta_\mu, r, \varphi)\,|_{r=R} = 0\,. \qquad (25)$$

The first step is also the separation of the spatial variables similar to (20). The spatial frequency variable and the transformation kernel are now functions of the integer values n and k.

$$K(n, k, r, \varphi) = \cos(n(\varphi - \varphi_0)) J_n\left(\mu_n k \frac{r}{R}\right) \qquad (26)$$

$$\beta_{nk}^4 = S^4 \left(\frac{\mu_{nk}}{R}\right)^2 \left[\left(\frac{\mu_{nk}}{R}\right)^2 - \frac{d_3 s - c^2}{S^4}\right] \quad (27)$$

$J_n(\cdot)$ is the Bessel function of order n, φ_0 is the phase of the initial excitation, and n and k are integers. To satisfy the boundary conditions (24,25) the variable μ_{nk} must be the k-th zero crossing of the Bessel function of order n.

The dependence on two independent integers is typical for 2D problems, but the sum in the inverse transformation and the discrete spatial frequencies $\beta_\mu = \beta_{nk}$ are not touched. The summation over μ results in a double sum over n and k.

The resulting transfer function model with an excitation function and an initial velocity is calculated according to (14). The inverse Laplace transformation of (14) is well known and the inverse SL transformation is realized by (19).

4.3. Discretization

To implement the analytic solution in the computer it has to be discretized. Therefore the deflection $z(x_1, x_2, t)$ is sampled at $t = mT_s$ with the temporal step size T_s. As shown in [4] a set of second order recursive systems can be derived, one for each spatial frequency β_μ.

To avoid aliasing the temporal frequencies may not be larger than half the sampling frequency [4]. So the summation in the inverse functional transformation is truncated to a finite length N.

5. RESULTS

Due to the physical interpretation of all the constants and variables the results can be interpreted very intuitively. Changes in the parameters cause changes in the wave propagation and therefore also in the produced sound as expected.

Every drummer knows about the importance of the hitpoint. An excitation in the middle of the membrane causes a very different sound than an excitation at $r_0 = \frac{2}{3}R$. Figure 1 illustrates this effect very clearly. There are much more vibration modes for $r_0 = \frac{2}{3}R$ in comparison to $r_0 = 0$. This effect is well known from the one-dimensional string model, but is even more impressive in two-dimensional membranes. Figure 2 shows the wave propagation of an impulse on a rectangular membrane.

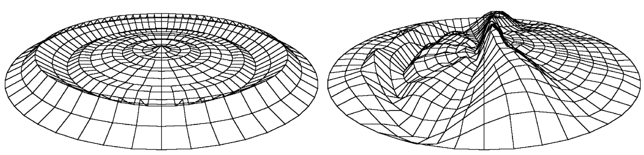

Fig. 1. Transversal vibrating membrane with excitation point at $r_0 = 0$ and $r_0 = \frac{2}{3}R$.

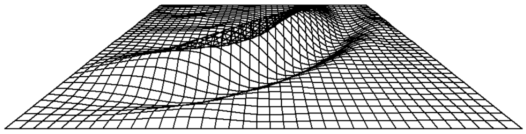

Fig. 2. Wave propagation on a rectangular membrane short-time after the excitation impulse.

6. CONCLUSIONS

The presented multidimensional transfer function model is a new approach for sound synthesis based on physical models. Even PDEs with multiple spatial dimensions can be handled by this method. The physics based multidimensional PDE can be transformed into the temporal and spatial frequency domain where it is solved analytically. After inverse transformation and discretization the solution can easily be implemented on a digital signal processor. Due to the physics based model the properties of the sound can be changed intuitively according to the experience of the musician.

7. REFERENCES

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