AN EFFICIENT ALGORITHM FOR FIR FILTER BANK COMPLETION

Corneliu Popeea[1], *Bogdan Dumitrescu*[2,1], *Boris Jora*[1]

[1] Department of Control and Computers
"Politehnica" University of Bucharest
313 Spl. Independenţei, 77206 Bucharest, Romania
e-mail: popeea,bogdan,jora@lucky.schur.pub.ro

[2] Tampere International Center for Signal Processing
Tampere University of Technology
P.O.Box 553, SF-33101 Tampere, Finland
e-mail: bogdand@cs.tut.fi

ABSTRACT

This paper presents an algorithm for designing an FIR paraunitary filter bank when one or several filters are given. The algorithm is based on the properties of the balanced state-space representation of the polyphase matrix. We show that this representation may be computed via a single RQ decomposition, thus gaining significant efficience with respect to previous work. Application of the algorithm to signal adapted filter banks is also discussed.

1. INTRODUCTION

We propose in this paper a new algorithm for computing the filters of a paraunitary FIR filter bank, given one or several of them. Some known solutions to this problem —named paraunitary embedding, or completion— will be shortly reviewed. Let $H_1(z)$ be the given filter (the case of $K > 1$ initial filters will be treated later), of order N; we denote $h_1 \in \mathcal{R}^{N+1}$ the corresponding vector of coefficients; let m be the number of channels of the filter bank. We aim to find filters $H_k(z)$, $k = 2 : m$, with impulse responses $h_k(i)$, $i = 0 : N$, such that to obtain an m-channel paraunitary filter bank. We suppose that the filters order is a multiple of m, i.e. $N + 1 = m(n + 1)$.

The polyphase decomposition of $H_k(z)$ is

$$H_k(z) = \sum_{\ell=1}^{m} z^{-(\ell-1)} H_{k,\ell}(z^m). \tag{1}$$

The associated polyphase matrix is

$$E(z) = [H_{k,\ell}(z)]_{k,\ell=1:m}. \tag{2}$$

The filter bank is paraunitary if the polyphase matrix $E(z)$ is paraunitary, i.e. $E(z)E^T(z^{-1}) = I$. Looking only at the first row of $E(z)$, the previous relation shows what are the conditions $H_1(z)$ must fulfill in order to be a valid data to the completion problem.

A first algorithm for paraunitary completion was given in [1], where the polyphase matrix was parameterized in lattice form. A different approach was proposed in [2]; it takes advantage of a balanced state-space representation of the polyphase matrix; this is also the basis of our algorithm, which improves on several aspects on [2]. Other papers, as [3] and [4], give parameterizations for the more theoretical case when the filters $H_k(z)$, $k = 2 : m$, may have orders higher than N. The linear phase case is treated in [4] and [5], the later paper offering also an interesting view, based on the singular value decomposition (SVD), to our problem.

The current paper is structured as follows. In the next section, we review the ideas from [2] and provide some details not mentioned there, mainly taken from [6]. In section 3, we present our algorithm; in very few words, our main contribution here is to compact the balancing transformation and an orthogonal matrix completion (done via SVD in [2]), into a single QR factorization; the algorithm provides also a lattice parameterization of the polyphase matrix. The case when $K > 1$ filters are given is also treated. Finally, in section 4, we discuss the application of our algorithm to the computation of signal adapted filter banks.

2. THE STATE-SPACE APPROACH

The algorithm suggested in [2] uses a well known property of paraunitary matrices. Let A, B, C, D be a minimal realization of a paraunitary matrix $E(z)$. Then, there exists a (nonsingular) transformation T such that the matrix

$$\begin{bmatrix} T^{-1}AT & T^{-1}B \\ CT & D \end{bmatrix} \tag{3}$$

is orthogonal. The new state-space representation of $E(z)$ is called *balanced* and T is the balancing transformation. The transformation T may be computed by solving a Lyapunov equation, as shown below.

In the completion problem, this result may be used for the first row only of $E(z)$, which is defined by the given filter $H_1(z)$. Denoting it

$$E_1(z) = [H_{11}(z)\ H_{12}(z)\ \dots\ H_{1m}(z)], \tag{4}$$

the paraunitarity condition is $E_1(z)E_1^T(z^{-1}) = 1$. We associate with $E_1(z)$ (which is the transfer matrix of a linear system with m inputs and one output) the observable state-space representation,

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{5}$$

$$B = \begin{bmatrix} h_1(m) & h_1(m+1) & \dots & h_1(2m-1) \\ h_1(2m) & h_1(2m+1) & \dots & h_1(3m-1) \\ \vdots & \vdots & & \vdots \\ h_1(nm) & h_1(nm+1) & \dots & h_1(N) \end{bmatrix} \tag{6}$$

$$C_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \tag{7}$$

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$$D_1 = \left[\begin{array}{cccc} h_1(0) & h_1(1) & \dots & h_1(m-1) \end{array}\right] \quad (8)$$

where the shift matrix A has order n. Remark that (A, B) is controllable, as the last row of B is nonzero (otherwise, N would have a smaller value), i.e. the realization is minimal.

To find the balancing transformation, we compute the (positive definite, since A is stable) solution P of the discrete matrix Lyapunov equation

$$P = APA^T + BB^T, \quad (9)$$

then we find the Cholesky decomposition of the matrix P, i.e. $L \in \mathcal{R}^{n\times n}$ upper triangular, such that $LL^T = P$.

Now, the balanced state-space representation of the polyphase matrix $E_1(z)$ is

$$\tilde{A} \leftarrow L^{-1}AL, \;\; \tilde{B} \leftarrow L^{-1}B, \;\; \tilde{C}_1 \leftarrow C_1 L. \quad (10)$$

In this form, following the result in the beginning of the section, the paraunitarity of $E_1(z)$ is expressed as the orthogonality of the rows of the $(n+1)\times(n+m)$ system matrix

$$S_1 = \left[\begin{array}{cc} \tilde{A} & \tilde{B} \\ \tilde{C}_1 & D_1 \end{array}\right], \quad (11)$$

i.e. $S_1 S_1^T = I$.

All the valid completions of the filter bank are obtained by completing the matrix S_1 with $m-1$ rows, such that an orthogonal matrix

$$S = \left[\begin{array}{cc} \tilde{A} & \tilde{B} \\ \tilde{C} & D \end{array}\right] \quad (12)$$

of size $(n+m)\times(n+m)$ is obtained, where

$$\tilde{C} = \left[\begin{array}{c} \tilde{C}_1 \\ \tilde{C}_2 \end{array}\right], \;\; D = \left[\begin{array}{c} D_1 \\ D_2 \end{array}\right]. \quad (13)$$

The valid completions may be parameterized in the form

$$S = \left[\begin{array}{cc} I_{n+1} & 0 \\ 0 & V^T \end{array}\right] \hat{S}, \quad (14)$$

where $\hat{S}$ is a particular completion, e.g. obtained via the QR factorization of S_1^T, and V is an arbitrary orthogonal matrix of size $(m-1)\times(m-1)$. We remind that in [2], the SVD was used to complete S_1; however, a simple QR factorization is sufficient, since the rows of S_1 are orthogonal, i.e. the completion problem is well conditioned.

All the state-space representations $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ of the form (12) belong to paraunitary matrices $E(z)$ expressed as (2). Therefore, the coefficients of the filters $H_k(z)$, $k = 2 : m$, are determined by computing the Markov coefficients $D, \tilde{C}\tilde{B}, \tilde{C}\tilde{A}\tilde{B}, \dots, \tilde{C}\tilde{A}^{n-1}\tilde{B}$, since

$$E(z) = D + \sum_{i=1}^{n} z^{-i}\tilde{C}\tilde{A}^{i-1}\tilde{B}, \quad (15)$$

and identifying appropriately the terms in (2) and (1).

If the matrix H is defined as containing on columns the impulse responses of the filters, i.e.

$$H = [h_1 \; h_2 \; \dots \; h_m], \quad (16)$$

then we remark that the parameterization (14) may be written equally well

$$H = \hat{H} \left[\begin{array}{cc} 1 & 0 \\ 0 & V \end{array}\right], \quad (17)$$

where $\hat{H}$ contains the filters given by $\hat{S}$ (and has the given h_1 on its first column).

Due to the form of A in (5), the Lyapunov equation (9) may be solved by a simple recursion. However, we will show in the next section that it is possible to compute directly the orthogonal completion (12) and the Cholesky factor L by a single QR factorization, starting from (5,6).

Also, the Markov coefficients (15) can be computed efficiently. For this, it is sufficient to apply the inverse of the transform (10), with the known L. We obtain A and B as in (5,6) (no computation required), and $C \leftarrow \tilde{C}L^{-1}$. Expressing input-output relations, the Markov coefficients are computed using the same (15) (with tildes removed), with the great advantage of having A as in (5).

3. THE NEW COMPLETION ALGORITHM

The relations defining the state-space transformation (10) may be written —using also (13) and denoting e_k the k-th unit vector— in the equivalent form

$$\left[\begin{array}{cc} C_1 L & D_1 \\ AL & B \end{array}\right] = \left[\begin{array}{cc} e_1^T & 0 \\ 0 & L \end{array}\right] \left[\begin{array}{cc} \tilde{C} & D \\ \tilde{A} & \tilde{B} \end{array}\right], \quad (18)$$

where the matrix

$$\tilde{Q} = \left[\begin{array}{cc} \tilde{C} & D \\ \tilde{A} & \tilde{B} \end{array}\right] \quad (19)$$

is orthogonal (as a simple permutation of S).

However, the balancing transformation L depends only on A and B, as immediate from (9). Therefore, we take from (18) only the part involving A and B, i.e.

$$\left[\begin{array}{cc} AL & B \end{array}\right] = \left[\begin{array}{cc} 0 & L \end{array}\right] Q, \quad (20)$$

where Q is an orthogonal matrix. As $\tilde{A}$ and $\tilde{B}$ are uniquely determined by A, B and L, relation (18) shows that the matrices Q and $\tilde{Q}$ have the last n rows equal, but otherwise may be different, i.e.

$$\tilde{Q} = \mathrm{diag}(Q_0, I_n)Q, \text{ with orthogonal } Q_0. \quad (21)$$

We begin by analyzing how Q can be computed. In relation (20), the matrix Q is orthogonal and the matrix $[0 \; L] \in \mathcal{R}^{n\times(m+n)}$ is right upper triangular. Therefore, (20) represents an RQ factorization. Due to the form (5) of the matrix A, the left term in (20) has the form, e.g. for $m = 3$, $n = 2$,

$$\left[\begin{array}{ccccc} 0 & \ell & b & b & b \\ 0 & 0 & b & b & b \end{array}\right], \quad (22)$$

where the elements of L and B are denoted generically by ℓ and b, respectively. Correspondingly, the matrix Q has the structure

$$\left[\begin{array}{ccccc} \tilde{c} & \tilde{c} & d & d & d \\ \tilde{c} & \tilde{c} & d & d & d \\ \tilde{c} & \tilde{c} & d & d & d \\ 0 & \tilde{a} & \tilde{b} & \tilde{b} & \tilde{b} \\ 0 & 0 & \tilde{b} & \tilde{b} & \tilde{b} \end{array}\right]. \quad (23)$$

Remark that the matrix $\tilde{A}$ is in Schur form and all its eigenvalues are zero, since we have here FIR filters. To have the complete picture of (20), we note that

$$\left[\begin{array}{cc} 0 & L \end{array}\right] = \left[\begin{array}{ccccc} 0 & 0 & 0 & \ell & \ell \\ 0 & 0 & 0 & 0 & \ell \end{array}\right]. \qquad (24)$$

The structure (22) shows that the RQ factorization (20) may be computed *without* knowing L from the beginning, although L appears in what is usually a constant matrix for an RQ factorization—the left term of (20). We denote $G = [AL\ B]$ and aim to find Householder reflectors $U_n, \ldots, U_1$ such that

$$GU_n \ldots U_1 \qquad (25)$$

is right upper triangular, like in (24), computing also L on the fly. Therefore, we will compute Q from (20), as

$$Q = U_1 \ldots U_n. \qquad (26)$$

As L is not known, G is initialized with $[0\ B]$. The successive transformations from (25) will be stored in place in G; finally, G will be right upper triangular, as in (24). The algorithm to be presented works on the rows of G, from bottom to top.

The last row in (22) depends only on B, that is we can compute a Householder reflector

$$U_n = \mathrm{diag}(I_n, \tilde{U}_n), \text{ with } \tilde{U}_n \in \mathcal{R}^{m\times m}, \qquad (27)$$

such that GU_n has only one nonzero element on its last row, i.e.

$$G(n, n+1:n+m)\tilde{U}_n = L(n,n)e_n^T. \qquad (28)$$

We have thus computed the last row of L, which has only one nonzero element, see (24).

As clear from (22), the row $n-1$ of G contains the row n of L, which is now available, thus we set $G(n-1,n) \leftarrow L(n,n)$. Then, we update the row $n-1$ of G with the previous reflector U_n, i.e.

$$G(n-1, n+1:n+m) \leftarrow G(n-1, n+1:n+m)\tilde{U}_n \qquad (29)$$

We notice that the element $G(n-1, n+m)$ will no more be touched by the RQ factorization, and thus $G(n-1, n+m) = L(n-1, n)$. We now compute the reflector

$$U_{n-1} = \mathrm{diag}(I_{n-1}, \tilde{U}_{n-1}, 1) \qquad (30)$$

such that the row $n-1$ of G has the necessary zero elements, obtaining also the next diagonal element of L, i.e.

$$G(n-1, n:n+m-1)\tilde{U}_{n-1} = L(n-1, n-1)e_n^T. \qquad (31)$$

For each row k, the algorithm follows the same pattern as above. First, the row $k+1$ of L (more precisely, $L(k+1, k+1:n)$, computed at the previous iteration) is inserted in row k of G, as $G(k, k+1:n)$. Then, the row k of G is updated with all the previous reflectors. Finally, the row is brought to the desired zero structure via an appropriate reflector having the structure

$$U_k = \mathrm{diag}(I_k, \tilde{U}_k, I_{n-k}). \qquad (32)$$

The algorithm is summarized in Fig. 1. The number of operations of this algorithm is exactly the same as for an usual RQ factorizations. (The order of the operations is changed with respect to the most used form of RQ factorization, where all leading rows are updated immediately after the computation of a reflector.)

The algorithm in Fig. 1 provides the orthogonal Q from (20), in the form (26). Notice that the first column of Q is e_1. We now want to compute $\tilde{Q}$ as in (18), i.e. to ensure $\tilde{C}_1$ as in (10). The RQ factorization from the algorithm in Fig. 1 may be continued with the reflector U_0 such that

$$\left[\begin{array}{cc} C_1 L & D_1 \\ AL & B \end{array}\right] Q^T U_0 = \left[\begin{array}{c} w^T \\ \hline 0 \quad L \end{array}\right] U_0 = \left[\begin{array}{cc} e_1^T & 0 \\ 0 & L \end{array}\right], \qquad (33)$$

where $w^T = \left[\begin{array}{cc} e_1^T L & D_1 \end{array}\right] Q^T$. As it should not touch L, the reflector U_0 for which $w^T U_0 = e_1^T$ must have the structure (21), i.e. $U_0 = \mathrm{diag}(\tilde{U}_0, I_n)$. It results that the rightmost n elements of w are already zero after the RQ factorization and update of the extra row $\left[\begin{array}{cc} C_1 L & D_1 \end{array}\right]$ (this condition may be used also to check the paraunitarity of the given filter $H_1(z)$). We have thus obtained the matrix $\tilde{Q}$ from (19), in the form

$$\tilde{Q} = U_0 Q = U_0 U_1 \ldots U_n, \qquad (34)$$

where the reflectors have the structure (32), confirming the structure of Q suggested in (23).

A lattice representation of the polyphase matrix $E(z)$ results from (34), in conjunction with the state-space representation (19). It is interesting to notice that a similar lattice has been proposed in the early paper [7,Fig.6], using only theoretical developments and without indicating an algorithm for its computation; also, Givens rotations were used instead of Householder reflectors. Due to the lack of space, we leave the interested reader to fill in the details, with the help of [7].

The case of $K > 1$ given filters is a simple extension of the case $K = 1$, due to the fact that the completion is essentially determined by a single filter (which determines uniquely the matrices $\tilde{A}$ and $\tilde{B}$). Denote $F_1 = [h_1 \ldots h_K]$ the matrix of the given impulse responses. If this is a valid set of filters, then $F_1^T F_1 = I$ (as well, for a paraunitary filter bank, $H^T H = I$). Let $\hat{H}$ be the filter bank obtained by the algorithm described in this section, starting from h_1. We must thus compute the matrix V from (17), where $H = [F_1\ F_2]$, and F_2 is the desired completion. Equivalently, we must compute a matrix $V_1 \in \mathcal{R}^{m\times(K-1)}$ with orthogonal columns such that

$$F_1 = \hat{H} \left[\begin{array}{cc} 1 & 0 \\ 0 & V_1 \end{array}\right]. \qquad (35)$$

This is an overdetermined system whose solution is

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & V_1 \end{array}\right] = (\hat{H}^T \hat{H})^{-1} \hat{H}^T F_1 = \hat{H}^T F_1. \qquad (36)$$

The matrix V_1 is completed to orthogonality (via QR factorization) in $V = [V_1\ V_2]$ and the completion F_2 results from (17). The freedom in choosing F_2 is reflected by the freedom in completing V with V_2.

4. SIGNAL ADAPTED FILTER BANKS

An application of the completion algorithm given in the previous section is at the computation of signal adapted filter banks, as suggested in [8]. Let $r(k)$, $k = 0:N$, be the autocorrelation sequence of the filter bank input signal and $R = \mathrm{Toep}(r(0), r(1), \ldots, r(N))$

0. Initialize $G(1:n, n+1:n+m) = B$, with B as in (6), and $G(1:n, 1:n) = 0$.
1. For $k = n : -1 : 1$ (compute row k of G)
   1.1. If $k < n$
      1.1.1. Insert previous row of L in current row of G: $G(k, k+1:n) \leftarrow G(k+1, k+m+1:n+m)$
      1.1.2. For $i = n : -1 : k+1$ (update current row with all previous reflectors)
         1.1.2.1. $G(k, i+1:i+m) \leftarrow G(k, i+1:i+m)\tilde{U}_i$
   1.2. Compute reflector $\tilde{U}_k \in \mathcal{R}^{m\times m}$ such that $G(k, k+1:m)\tilde{U}_k$ is zero excepting the last element

Output: $L = G(1:n, n+1:n+m)$ upper triangular, $Q^T = U_n \ldots U_1$, with U_k as in (32).

Fig. 1. Adapted RQ factorization algorithm for computing L and Q from (20).

the corresponding Toeplitz matrix. For an analysis filter bank defined by (16), the energy of the signals at the output of each filter $H_k(z)$ is $\sigma_k^2 = h_k^T R h_k$. The coding gain of the filter bank is

$$G_c(H) = \frac{\frac{1}{m}\sum_{k=1}^{m}\sigma_k^2}{\left(\prod_{k=1}^{m}\sigma_k^2\right)^{(1/m)}} = \frac{1}{\left(\prod_{k=1}^{m}\sigma_k^2\right)^{(1/m)}}. \quad (37)$$

We want to design a (paraunitary) filter bank H such that the coding gain is maximized. This is a nonconvex optimization problem, for which there is no algorithm to ensure optimality of the solution. The heuristic proposed in [8] is to design first the filter h_1 maximizing σ_1^2, i.e. an optimum compaction filter; to this purpose, we proposed an efficient algorithm in [9], based on semidefinite programming, always ensuring optimality, at relatively low cost.

Then, the other filters are found. Computationally, this is cheaper than finding the first filter. Suppose that, starting with the optimal h_1, the algorithm in the previous section computes a completion such that $\hat{H}$ is paraunitary and has h_1 on the first column. Then, all the valid completions are given by (17). Denoting

$$\hat{R} = \hat{H}^T R \hat{H}, \quad (38)$$

it results from the optimality of h_1 that

$$\hat{R} = \mathrm{diag}(\sigma_1^2, R_2). \quad (39)$$

(Otherwise σ_1^2 could be increased via an orthogonal transform, as R is positive definite.) Now, the filter bank optimizing (37) *given* h_1 is defined by V from the eigendecomposition

$$V^T \hat{R} V = \Lambda, \quad (40)$$

with diagonal Λ. It results that $H^T R H$ is diagonal. The proof of these facts may be found in [6].

This two-step algorithm is not giving the optimum coding gain, but often a very good approximation. Experimental data are presented in [6], available also on the web.

5. REFERENCES

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