RECURSIVE SINGLE FREQUENCY ESTIMATION

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ABSTRACT

A new divide-and-conquer method for estimating the frequency of a single complex sinusoid in additive uncorrelated noise is proposed. Its computational complexity is comparable to previous fast methods (roughly 2N complex multiplies and log2(N) arctangents for N a power of 2). However, it nearly achieves the Cramer-Rao bound for a wider range of input frequency and signal-to-noise-ratio (SNR) values. Simulations are presented to demonstrate its performance.

1. INTRODUCTION

A common problem in communications is the estimation of the frequency of a single complex sinusoid in additive noise. Since the wireless environment is often characterized by poor SNR, it is of interest to find methods of frequency estimation that work well in low SNR environments.

Previous work in this area includes that of [1], [2] and their references. All of the methods treated in [1] fail below different SNR thresholds. Hence it would be useful to increase the SNR before applying these methods.

This is the approach taken in [2]. However, the price for decreasing the SNR threshold is that the range of frequency over which the estimator is valid is reduced.

In this paper, first a method is described for preprocessing a sample of sinusoid+noise data to increase its SNR. This preprocessed data can then be used as input to any one of a number of frequency estimation methods. The accuracy of the resulting frequency estimates will be improved due to better SNR at their input. Also, there is no reduction in the valid frequency range of the estimate.

Then, extending the basic SNR improvement idea, a recursive method using approximately $2N$ complex multiplies, $5\log_2 N$ real multiplies, $\log_2 N$ inverse square roots and $\log_2 N$ arctangent operations is presented for $N = 2^M$. This technique nearly achieves the Cramer-Rao bound on estimation error variance over a wider range of SNR than previous algorithms of similar complexity; in fact, its performance is comparable to the much more computationally intensive maximum likelihood estimator (MLE). The reduction in computation relative to the MLE is especially dramatic for large N. The technique can also be used to lesser advantage for $N = 2^J * L$.

2. SNR IMPROVEMENT

Assume the following model for the data:

$$x_n = Ae^{j(\omega n+\phi)} + \upsilon_n, \quad n = 0,..,N-1 \qquad (1)$$

(we only consider even values of N here). The noise sequence υ_n is assumed to be complex, zero-mean and uncorrelated with variance σ_υ^2. The SNR is then given by A^2/σ_υ^2.

To increase the SNR prior to frequency estimation, consider

$$z_n = x_n + e^{j\theta} x^*_{N-1-n}, \quad n = 0,...,N-1 \qquad (2)$$

Hence

$$z_n = A\left(e^{j(\omega n+\phi)} + e^{j(\omega n - \omega(N-1) - \phi + \theta)}\right) + \varepsilon_n \qquad (3)$$

where

$$\varepsilon_n = \upsilon_n + e^{j\theta}\upsilon^*_{N-1-n} \qquad (4)$$

Since the samples of $\{\upsilon_n\}$ are uncorrelated, the variances of υ_n and $e^{j\theta}\upsilon^*_{N-1-n}$ will add and ε_n will have 3dB more power than υ_n. However, if $\theta = [\omega(N-1)+2\phi]_{\mathrm{mod}\,2\pi}$, the sinusoid and its conjugate time-reverse add coherently, increasing the sinusoidal power by 6dB. This would increase the SNR by 3dB without knowledge of ω.

Note that θ can be estimated by maximizing the power of $\{z_n\}$, namely

$$\sum_{n=0}^{N-1} |z_n|^2 = 2\sum_{n=0}^{N-1}|x_n|^2 + 4\,\mathrm{Re}\sum_{n=0}^{N/2-1} e^{-j\theta} x_n x_{N-1-n} \qquad (5)$$

with respect to θ. The solution to this is given by

$$\theta = \arg\left[\sum_{n=0}^{N/2-1} x_n x_{N-1-n}\right] \quad (6)$$

(the arg function used here gives values in $\{-\pi,\pi\}$).

Using this value in (2) yields $\{z_n\}$ with SNR roughly 3dB better than $\{x_n\}$. Note that for (2) it is sufficient to compute $\cos\theta$ and $\sin\theta$, i.e.

$$\cos\theta = \frac{A}{\sqrt{A^2+B^2}} \quad \text{and} \quad \sin\theta = \frac{B}{\sqrt{A^2+B^2}} \quad (7)$$

where A and B are the real and imaginary parts of the dot product in (6).

The sequence $\{z_n\}$ can then be used as input in any of a number of methods for determining the frequency. In particular, consider the two types of estimators described by Kay in [1]. The ones considered are variants of the "phase average" (PA) estimator

$$\omega \approx \sum_{n=0}^{N-2} w_n \arg(x_{n+1}x_n^*) \quad (8)$$

and the "linear predictive" (LP) estimator

$$\omega \approx \arg(\sum_{n=0}^{N-2} w_n x_{n+1}x_n^*) \quad (9)$$

which use different choices of the symmetric weight sequence $\{w_n, n=0,N-2\}$, with $w_n = w_{N-2-n}$.

Equation (2) introduces a symmetry into $\{z_n\}$ that can be used to simplify these computations. It can be easily shown that

$$z_{N-1-n} = e^{j\theta} z_n^* \quad (10)$$

It follows that

$$z_{N-1-n} z_{N-2-n}^* = z_{n+1} z_n^*, \quad n = 0,\ldots,N-2 \quad (11)$$

Hence if the preprocessing is done before application of (8) or (9), they become

$$\omega \approx 2\sum_{n=0}^{N/2-2} w_n \arg(z_{n+1}z_n^*) + w_{N/2-1}\arg(z_{N/2}z_{N/2-1}^*)$$

and (12)

$$\omega \approx \arg(2\sum_{n=0}^{N/2-2} w_n z_{n+1}z_n^* + w_{N/2-1} z_{N/2} z_{N/2-1}^*) \quad (13)$$

respectively.

3. RECURSIVE SNR ENHANCEMENT

In addition to increasing the SNR by 3dB, application of (2) produces the redundancy in $\{z_n\}$ given by (10). Thus, the number of independent points is half as large as in (1). Also, the functional form of $\{z_n, n=0,N/2-1\}$ is identical to that of (1) and the underlying sinusoid has the same frequency as $\{x_n\}$. This is true even if the estimate of θ is not perfect, since an error in θ will affect only the initial phase and amplitude of the resulting sinusoid, not its frequency.

This suggests applying the algorithm again to $\{z_n, n=0,N/2-1\}$ in a divide-and-conquer fashion to get $\{z'_n, n=0,N/4-1\}$, then again to get $\{z''_n, n=0,N/8-1\}$, etc., until $N/2^m$ is odd. Ideally, each iteration increases the SNR of the remaining points by 3dB (this doesn't happen at every iteration in practice due to errors in the θ estimates). The original-length sinusoid can then be reconstructed with improved SNR using (10) iteratively, after which one of the algorithms of [1] or [2] can be applied to the reconstructed $\{x_n, n=0,N-1\}$ to determine the sinusoid frequency.

To maximize the SNR improvement of this method, let N be a power of 2, $N=2^M$. Define $z_n^{(0)} \equiv x_n, n=0,\ldots,N-1$, and perform (14) and (15) for $k=0$ to $M-1$:

$$\theta_k = \arg\left[\sum_{n=0}^{N/2^{k+1}-1} z_n^{(k)} z_{N/2^k-1-n}^{(k)}\right] \quad (14)$$

$$z_n^{(k+1)} = z_n^{(k)} + e^{j\theta_k} z_{N/2^k-1-n}^{(k)*},$$

$$n = 0,\ldots,N/2^{k+1}-1 \quad (15)$$

Note that there is only one point in the M th sequence, $z_0^{(M)}$. We can now reconstruct the original sinusoid *with reduced noise* by using (10), i.e.

$$x_0^{(M)} = z_0^{(M)} \quad (16)$$

and performing the following two steps for k decreasing from M to 1:

$$x_n^{(k-1)} = x_n^{(k)}, \qquad n=0,\ldots,N/2^k-1 \quad (17)$$

$$x_{N/2^{k-1}-1-n}^{(k-1)} = e^{j\theta_{k-1}} x_n^{(k)*}, \qquad n=0,\ldots,N/2^k-1 \quad (18)$$

The final sequence $\{x_n^{(0)}, n=0,N-1\}$ is a version of the original sinusoid with reduced noise. The algorithms from [1] and [2] can then be applied to this sequence as above.

Note that all information defining the reduced-noise sequence $\{x_n^{(0)}\}$ is contained in the combination of $\left|z_0^{(M)}\right|$, $\arg(z_0^{(M)})$ and $\{\theta_k, k=0,\log_2(N)-1\}$, that is, in only $\log_2(N)+2$ real numbers.

4. APPLICATION TO KAY'S ESTIMATOR

The PA estimators of the form (8) treated in [1] require only the principal values of the phase differences between adjacent samples to compute their results. It turns out that, using (16)-(18) these phase differences for $\{x_n^{(0)}, n=0,N-1\}$ can easily be constructed directly from

$\arg(z_0^{(M)})$ and $\{\theta_k\}$ without the complex multiplies of (18), as follows.

Let $\varphi = \arg(z_0^{(M)})$. Then

$$\arg(x_1^{(M-1)} x_0^{(M-1)}*) = princ(\theta_{M-1} - 2\varphi) \quad (19)$$

where the principal value function $princ(\cdot)$ maps its input (assumed to be a phase) to the interval $\{-\pi,\pi\}$. Let $\{\Phi_k(n), n = 0, N/2^k - 2\}$ be the phase difference sequence for $\{x_n^{(k)}\}$. That is, (19) defines the one-point sequence $\{\Phi_{M-1}(0)\}$. Then (11), (17) and (18) yield the step-down recursion:

$$\{\Phi_{k-1}\} = \{\Phi_k \mid princ(\theta_{k-1} - 2\theta_k + 2\varphi) \mid \Phi_k\},$$
$$k = M-1, M-2, \ldots, 1 \quad (20)$$

where "|" denotes sequence concatenation.

Figure 1 shows an illustrative example of the reconstructed phase difference Φ_0 for this algorithm for a realization of the case N=128, f=0.125Hz, SNR=10dB. For comparison, the phase difference for a noiseless version of this signal would be a constant value of π/4 = 0.7854 rad.

Using (20), the Kay estimator becomes

$$\omega \approx 2\sum_{n=0}^{N/2-2} w_n \Phi_0(n) + w_{N/2-1}\Phi_0(N/2-1) \quad (21)$$

However, since there are only M distinct phase difference values in Φ_0, this can be simplified. If

$$\psi_k = princ(\theta_k - 2\theta_{k+1} + 2\varphi), \quad k = 0,\ldots,M-2 \quad (22)$$

and $$\psi_{M-1} = princ(\theta_{M-1} - 2\varphi), \quad (23)$$

it can be shown that the optimal $\{w_n\}$ from [1] gives

$$\omega \approx \sum_{k=0}^{M-1} c_k \psi_k \quad (24)$$

where $$c_k = \frac{N}{N^2-1}\frac{2^{2k+1}+1}{2^{k+1}} \quad (25)$$

(proof omitted due to lack of space). Hence (21) can be computed with only $\log_2 N$ real multiplies.

The complexity of this complete recursive frequency estimation algorithm consists of

a) $2(N-1)$ complex multiplies to compute (14) and (15) (since $N/2 + N/4 + \ldots + 1 = N-1$),

b) $\log_2 N$ inverse square roots for (7),

c) $5\log_2 N$ real multiplies for (7) and (21) and

d) $\log_2 N + 1$ arctan's for (14) and computation of φ.

By comparison, the PA method of [2] requires roughly $N-K$ complex multiplies, $N-K$ real multiplies and $N-K$ arctan's, where $K << N$ is a parameter. The PA method of [1] uses $K = 1$.

It is possible to use other properties of the complex sinusoid to derive other versions of this recursive single frequency estimator, e.g. decimation-in-frequency and decimation-in-time versions with and without the conjugate time reversal in (2). Space limitations do not permit discussion of these here. They will be presented elsewhere. However, their performance is similar to the algorithm presented here.

Note that for $N = 2^J * L$, it is possible to apply J stages of SNR enhancement to the data. The resulting L-length sequence may have SNR as much as $3J$ dB better than the original. This sequence can be expanded back to its original length using (10) as above. Finally, the algorithms of [1] or [2] may be applied to the result, giving an improved frequency estimate.

5. COMPUTER SIMULATIONS

MATLAB simulations show that this method very nearly achieves the Cramer-Rao bound for SNR levels comparable to the classical MLE estimator (i.e. the DFT peak location). This is much better than existing algorithms of this complexity and useful frequency range.

How much better depends on N and the frequency ω. Figure 2 shows $10\log_{10}$(1/error variance) vs. SNR, for N=32 and $f = \omega/2\pi$ = 0.05Hz. The SNR threshold is reduced from 9dB for the PA algorithm [1] to 1dB for the new recursive PA (RPA) algorithm. Also, note that in this case the variances of both the RPA and recursive LP (RLP) estimators are lower than the MLE for SNR < 7dB.

For N=128 and f = 0.2Hz, the SNR threshold is reduced from 11dB for the PA algorithm to -3dB for the RPA algorithm, as shown in figure 3.

In principle, all the methods in [1], the RPA and RLP are valid for the range $\omega = \{-\pi,\pi\}$. However, those discussed in [2] are limited to a maximum total range of π radians or less. In fact, both the PA and RPA methods work best in the range $\{-\pi/2, \pi/2\}$; performance degrades outside this range. This is illustrated by the results for N=32 and f = 0.45Hz shown in figure 4.

Interestingly, neither the RLP algorithm nor the LP of [1] is much affected by the frequency, though the RLP algorithm performs better than the LP algorithm over frequency and SNR.

It is relatively easy to modify the RPA algorithm to work reliably over the entire frequency range $\{-\pi,\pi\}$. Description of this modification is omitted due to space limitations and will be presented elsewhere.

6. CONCLUSIONS

A divide-and-conquer method for the estimation of the frequency of a single sinusoid in additive noise has been proposed. In full precision, it has performance

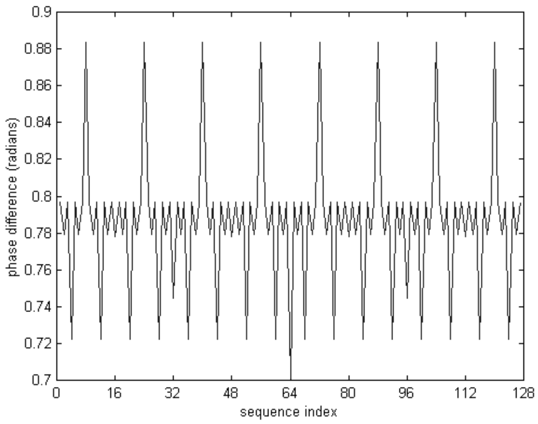

Fig. 1. Typical reconstructed phase difference sequence derived from (20); f=0.125Hz, N=128, SNR = 10dB; for a noiseless sinusoid, this would be a constant value of $\pi / 4$ = 0.7854

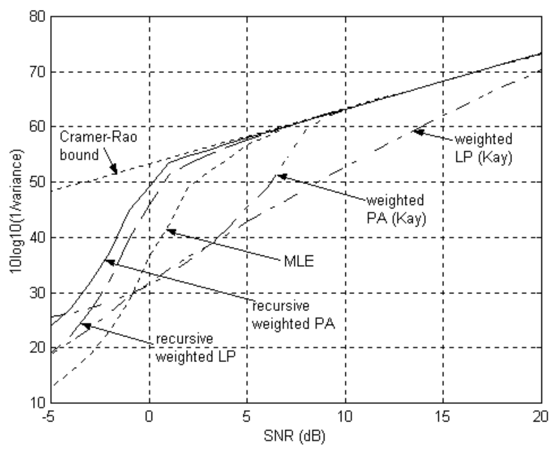

Fig. 2. Comparison of Kay algorithms with results for the same algorithms applied to recursively enhanced data for N=32, f=0.05Hz, 25000 realizations

comparable to the MLE over a comparable range of SNR without restricting the range of ω for which the estimate is useful. Its computational complexity is lower than the fastest methods of [1] and [2] (considering that the arctan operation on practical DSPs has a higher cycle count than a complex multiply). In spite of this, the SNR threshold of the new algorithm is lower than the fast methods of [1] and [2] that are valid over a comparable frequency range.

The recursive nature of the new algorithm raises the question of how well it performs in fixed-point arithmetic. Certainly errors at early stages will propagate, so care must be taken to maintain maximum precision at each iteration. However, this should be manageable. Also, note that we are using the $\{\theta_k\}$ and φ directly in (22)-(25) and are not adding error by computing (18). Hence the numerical properties of the complete algorithm

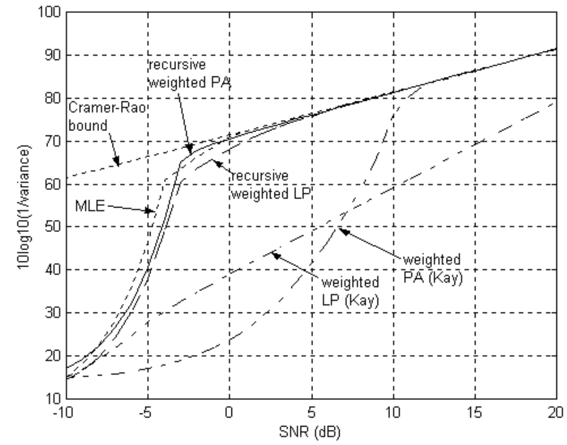

Fig. 3. Same as figure 2 for N=128, f=0.20Hz, 50000 realizations

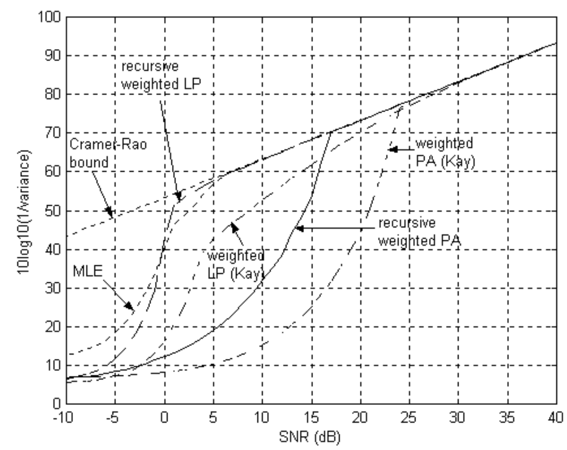

Fig. 4. Same as figure 2 for N=32, f=0.45Hz, 25000 realizations

are expected to be better than the SNR enhancement algorithm described in (14)-(18), in addition to requiring less computation, at least for estimators of the form of (8).

7. REFERENCES

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