STABILITY OF THE 2-D FORNASINI-MARCHESINI MODEL WITH PERIODIC COEFFICIENTS

Tamal Bose

Electrical & Computer Engineering
Utah State University
Logan, UT 84322-4120

R. Thamvichai and M. Radenkovic

Electrical & Computer Engineering
University of Colorado
Boulder, CO 80309

ABSTRACT

The stability of two-dimensional (2-D) periodically shift varying (PSV) filters is considered. These filters have applications in filtering video signals with cyclostationary noise, image and video scrambling, and design of multiplierless filters. The considered system is represented in state space by the first model of Fornasini-Marchesini with periodic coefficients. Then the stability of this model is studied. Two necessary conditions and two sufficient conditions are established for asymtotic stability. The conditions are easy to use and computationally simple.

1. INTRODUCTION

Two-dimensional (2-D) periodically shift varying (PSV) filters have many applications in engineering, such as in processing digital video with cyclostationary noise, in designing image scramblers, and in the design of 2-D multirate filter banks. PSV filters are also important for designing 2-D filters with power-of-two coefficients[2]. These multiplierless filters are extremely useful for real-time processing of large amounts of data. During the past decade, 2-D PSV filters have been analyzed and designed in [3]-[7]. In [3], 2-D PSV filters have been analyzed in direct form. In [4], equivalent shift invariant structures were derived for 2-D PSV filters. The stability of 2-D PSV filters have been analyzed to some extent. Stability of PSV filters has also been analyzed by finding equivalent shift-invariant structures [5],[6]. In [7], PSV filters formulated as the second model of Fornasini-Marchesini were considered for stability and some conditions and properties were established.

In this paper, 2-D PSV filters that are formulated as the first model of Fornasini-Marchesini [8] are considered. This model is more general since it has three state matrices in the state equation as opposed to two for the second model. Also, this model leads to the well known Attasi model [9] which has found applications in the design of separable 2-D filters. The considered filter is then analyzed for stability using Lyapunov energy functions. Some conditions are then established for stability.

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2. SYSTEM DESCRIPTION

The difference equation representation for a linear 2-D PSV system can be written in the following form:

$$y(i,j) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} a_{mn}(i,j)y(i-m,j-n) + \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} b_{mn}(i,j)u(i-m,j-n) \quad (1)$$

where $(m,n) \neq (0,0)$ for a_{mn}. The coefficients are periodically shift variant, i.e., $a_{mn}(i,j) = a_{mn}(i+P,j) = a_{mn}(i,j+Q)$ and $b_{mn}(i,j) = b_{mn}(i+P,j) = b_{mn}(i,j+Q)$, where the period is (P,Q) and P,Q are positive integers, not both zero.

Like LSIV system, several different state-space forms with periodic coefficient matrices can be used to represent the above 2-D PSV system. The first model of Fornasini-Marchesini [8] is given below:

$$\begin{aligned} x(h+1,k+1) &= A_1(h,k)x(h,k+1) + \\ &\quad A_2(h,k)x(h+1,k) + \\ &\quad A_0(h,k)x(h,k) + \\ &\quad B(h,k)u(h,k) \\ y(h,k) &= C(h,k)x(h,k) \end{aligned} \quad (2)$$

where $x(h,k) \in R^{L\times 1}$, $u(h,k), y(h,k) \in R^{1\times 1}$, $A_0(h,k)$, $A_1(h,k)$, $A_2(h,k) \in R^{L\times L}$, $B(h,k) \in R^{L\times 1}$, and $C(h,k) \in R^{1\times L}$. The coefficient matrices, $A_0(h,k)$, $A_1(h,k)$, $A_2(h,k)$, are functions of $a_{mn}(i,j)$ and $b_{mn}(i,j)$ in (1) and periodically shift variant with period (P,Q). The initial conditions are assumed such that $x(i,j) = 0$, $i < 0$ or $j < 0$, and $x(i,0) = 0$, $x(0,j) = 0$, for $i > I$, $j > J$. In this paper, the zero-input stability of the state equation (2) is studied, and therefore, only the periodic coefficient matrices are considered.

3. DEFINITIONS AND OBSERVATION

Definition 1 (2-D system stability): A 2-D state-space system with state variable $x(i,j)$ is asypmtotically stable if $lim_{i+j\to\infty} x(i,j) = 0$.

Definition 2 (2-D energy function on horizontal and vertical direction): For a 2-D system with state variable $x(i,j)$, the energy in the horizontal direction is

$$H(k) = \sum_{l=0}^{\infty} ||\, x(l,k)\, ||$$

and the energy in the vertical direction is

$$V(h) = \sum_{l=0}^{\infty} ||\, x(h,l)\, ||$$

where $||.||$ is any vector norm.

Observation 1: If

$$lim_{k\to\infty} H(k) = 0 \quad and \quad lim_{h\to\infty} V(h) = 0,$$

then the system is stable.

Observation 1 is true because if the energy in both directions goes to zero, then the state variable $x(i,j)$ must also approach zero in any direction in the (i,j) plane.

4. STABILITY OF 2-D PSV SYSTEM

In this section, we begin by establishing a sufficient condition for the stability of the 2-D PSV filter.

Theorem 1 *Consider the zero-input 2-D PSV state space model given in* (2).

For $n = 0,1,2,...,Q-1$, *define*

$$F_n = \frac{\max_{\forall l}\left(\begin{array}{c} |||\, A_2((l-1)_{\mathrm{mod}\, P}, n)||| \\ +|||\, A_0(l_{\mathrm{mod}\, P}, n)||| \end{array}\right)}{1 - \max_{\forall l}(\, |||\, A_1(l_{\mathrm{mod}\, P}, n)|||\,)} \tag{3}$$

and for $m = 0,1,2,...,P-1$, *define*

$$G_m = \frac{\max_{\forall l}\left(\begin{array}{c} |||\, A_1(m,(l-1)_{\mathrm{mod}\, Q})||| \\ +|||\, A_0(m, l_{\mathrm{mod}\, Q})||| \end{array}\right)}{1 - \max_{\forall l}(\, |||\, A_2(m, l_{\mathrm{mod}\, Q})|||\,)} \tag{4}$$

where $|||.|||$ *is any matrix norm.*

If $|||\, A_1(h,k)||| < 1$, $|||\, A_2(h,k)||| < 1$ *for all* (h,k), *and*

$$\prod_{n=Q-1}^{0} F_n < 1 \text{ and } \prod_{m=P-1}^{0} G_m < 1, \tag{5}$$

then the system is asymptotically stable.

Proof: For the horizontal direction, using the system description and the initial conditions, we can write the state variables on the k-th horizontal line as

$$\begin{aligned} x(0,k) &= A_2(-1,k-1)x(0,k-1) \\ &= A_2(P-1,k-1)x(0,k-1), \end{aligned}$$

$$\begin{aligned} x(1,k) &= A_1(0,k-1)x(0,k)+A_2(0,k-1)x(1,k-1) \\ &\quad + A_0(0,k-1)x(0,k-1), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} x(i,k) &= A_1((i-1)_{\mathrm{mod}\, P},k-1)x(i-1,k) \\ &\quad +A_2((i-1)_{\mathrm{mod}\, P},k-1)x(i,k-1) \\ &\quad +A_0((i-1)_{\mathrm{mod}\, P},k-1)x(i-1,k-1), \end{aligned}$$

$$\vdots$$

Taking any vector norm on both sides and using the triangular inequality of norms, gives

$$||x(0,k)|| \leqslant |||A_2(-1,k-1)|||.||x(0,k-1)||,$$

$$\begin{aligned} ||x(1,k)|| \leqslant\ & |||A_1(0,k-1)|||.||x(0,k)|| \\ & +|||A_2(0,k-1)|||.||x(1,k-1)|| \\ & +|||A_0(0,k-1)|||.||x(0,k-1)||, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} ||x(i,k)|| \leqslant\ & |||A_1((i-1)_{\mathrm{mod}\, P},k-1)|||.||x(i-1,k)|| \\ & +|||A_2((i-1)_{\mathrm{mod}\, P},k-1)|||.||x(i,k-1)|| \\ & +|||A_0((i-1)_{\mathrm{mod}\, P},k-1)|||.||x(i-1,k-1)||, \end{aligned}$$

$$\vdots$$

Then summing up the above equations and transposing similar terms, we get

$$\begin{aligned} & (1-|||A_1(0,k-1)|||).||x(0,k)|| + \\ & (1-|||A_1(1,k-1)|||)\, .||x(1,k)|| + \dots \\ & (1-|||A_1((i-1)_{\mathrm{mod}\, P},k-1)|||)\, .||x(i-1,k)|| + \dots \end{aligned}$$

$$\begin{aligned} \leqslant & \left(\begin{array}{c} |||A_2(-1,k-1)||| \\ +|||A_0(0,k-1)||| \end{array}\right).||x(0,k-1)|| + \\ & \left(\begin{array}{c} |||A_2(0,k-1)||| \\ +|||A_0(1,k-1)||| \end{array}\right).||x(1,k-1)|| + \dots \\ & \left(\begin{array}{c} |||A_2((i-1)_{\mathrm{mod}\, P},k-1)||| \\ +|||A_0(i_{\mathrm{mod}\, P},k-1)||| \end{array}\right).||x(i,k-1)|| + \dots \end{aligned}$$

The above can be written in closed form as

$$\sum_{l=0}^{\infty} (1 - |||A_1(l_{\mathrm{mod}\, P}, k-1)|||)\, ||x(l,k)|| \leqslant \sum_{l=0}^{\infty} \left(\begin{array}{c} |||A_2((l-1)_{\mathrm{mod}\, P}, k-1)||| \\ +|||A_0(l_{\mathrm{mod}\, P}, k-1)||| \end{array}\right) ||x(l,k-1)||. \tag{6}$$

The left-hand side can be written as

$$\min_{\forall l} (1 - |||A_1(l_{\mathrm{mod}\,P}, k-1)|||) \sum_{l=0}^{\infty} ||x(l,k)||$$

$$\leq \sum_{l=0}^{\infty} (1 - |||A_1(l_{\mathrm{mod}\,P}, k-1)|||) \, ||x(l,k)||. \qquad (7)$$

Similarly, the right-hand side of (6) can be written as,

$$\sum_{l=0}^{\infty} \left(\begin{array}{c} |||A_2((l-1)_{\mathrm{mod}\,P}, k-1)||| \\ +|||A_0(l_{\mathrm{mod}\,P}, k-1)||| \end{array} \right) ||x(l, k-1)|| \leq$$

$$\max_{\forall l} \left(\begin{array}{c} |||A_2((l-1)_{\mathrm{mod}\,P}, k-1)||| \\ +|||A_0(l_{\mathrm{mod}\,P}, k-1)||| \end{array} \right) \sum_{l=0}^{\infty} ||x(l, k-1)||. \qquad (8)$$

Combining (6), (7), and (8), we get

$$\min_{\forall l} (1 - |||A_1(l_{\mathrm{mod}\,P}, k-1)|||) \sum_{l=0}^{\infty} ||x(l,k)|| \leq$$

$$\max_{\forall l} \left(\begin{array}{c} |||A_2((l-1)_{\mathrm{mod}\,P}, k-1)||| \\ +|||A_0(l_{\mathrm{mod}\,P}, k-1)||| \end{array} \right) \sum_{l=0}^{\infty} ||x(l, k-1)||. \qquad (9)$$

Defining $H(k) = \sum_{l=0}^{\infty} || \, x(l,k) \, ||$,(9) can be written as

$$\min_{\forall l} (1 - |||A_1(l_{\mathrm{mod}\,P}, k-1)|||) \, H(k)$$

$$\leq \max_{\forall l} \left(\begin{array}{c} |||A_2((l-1)_{\mathrm{mod}\,P}, k-1)||| \\ +|||A_0(l_{\mathrm{mod}\,P}, k-1)||| \end{array} \right) H(k-1).$$

The above is equivalent to

$$\left(1 - \max_{\forall l} (|||A_1(l_{\mathrm{mod}\,P}, k-1)|||) \right) H(k)$$

$$\leq \max_{\forall l} \left(\begin{array}{c} |||A_2((l-1)_{\mathrm{mod}\,P}, k-1)||| \\ +|||A_0(l_{\mathrm{mod}\,P}, k-1)||| \end{array} \right) H(k-1).$$

Since $|||A_1(i,j)||| < 1$ for all (i,j),

$$H(k) \leq \frac{\max_{\forall l} \left(\begin{array}{c} |||A_2((l-1)_{\mathrm{mod}\,P}, k-1)||| \\ +|||A_0(l_{\mathrm{mod}\,P}, k-1)||| \end{array} \right)}{\left(1 - \max_{\forall l} (|||A_1(l_{\mathrm{mod}\,P}, k-1)|||) \right)} H(k-1)$$

Now using definition (3), the above becomes

$$H(k) \leq F_{k-1} H(k-1).$$

Since all coefficient matrices are periodic (P,Q), we can write the above inequality as

$$H(k) \leq F_{(k-1)\,\mathrm{mod}\,Q} H(k-1). \qquad (10)$$

Let $k = r \; mod \, Q$ and using the recursion in (10), we have

$$H(k) \leq \left(\begin{array}{c} \left(\prod_{l=r-1}^{0} F_l \right) \left(\prod_{l=Q-1}^{0} F_l \right)^{n-1} \\ \left(\prod_{l=Q-1}^{(r-R)\,\mathrm{mod}\,Q} F_l \right) H(k - (nQ + R)) \end{array} \right) \qquad (11)$$

with $R \leq r$ and $\prod_{l=-1}^{0} F_l = 1$ for all integer n and R.

For the vertical direction, a similar approach can be used to obtain

$$V(h) \leq G_{(h-1)\,\mathrm{mod}\,P} V(h-1), \qquad (12)$$

where $V(h)$ is the energy function defined as $V(h) = \sum_{l=0}^{\infty} || \, x(h,l) \, ||$. Then using the recursion in (12) with $h = r \; mod \, P$, we have

$$V(h) \leq \left(\begin{array}{c} \left(\prod_{l=r-1}^{0} G_l \right) \left(\prod_{l=P-1}^{0} G_l \right)^{n-1} \\ \left(\prod_{l=P-1}^{(r-R)\,\mathrm{mod}\,P} G_l \right) V(h - (nP + R)) \end{array} \right) \qquad (13)$$

with $R \leq r$ and $\prod_{l=-1}^{0} G_l = 1$ for all integers n and R. From (11) and (13), we can see that if

$$\prod_{l=Q-1}^{0} F_l < 1 \text{ and } \prod_{l=P-1}^{0} G_l < 1$$

then clearly $lim_{k \to \infty} H(k) = 0$ and $lim_{h \to \infty} V(h) = 0$. This completes the proof. ∎

Theorem 2 *Consider the zero-input 2-D PSV state space model given in* (2)*. The system is stable if the following conditions are satisfied:*

1) $||| \, A_1(h,k)||| < 1$, $||| \, A_2(h,k)||| < 1$ *for all* (h,k),

2)$\max_{\forall (l,k)} |||A_1(l_{\mathrm{mod}\,P}, (k-1)_{\mathrm{mod}\,Q},)||| +$
$\max_{\forall (l,k)} \left(\begin{array}{c} |||A_2((l-1)_{\mathrm{mod}\,P}, (k-1)_{\mathrm{mod}\,Q})||| \\ +|||A_0(l_{\mathrm{mod}\,P}, (k-1)_{\mathrm{mod}\,Q})||| \end{array} \right) < 1,$
and

3)$\max_{\forall (h,l)} |||A_2((h-1)_{\mathrm{mod}\,P}, l_{\mathrm{mod}\,Q})||| +$
$\max_{\forall (h,l)} \left(\begin{array}{c} |||A_1((h-1)_{\mathrm{mod}\,P}, (l-1)_{\mathrm{mod}\,Q})||| \\ +|||A_0((h-1)_{\mathrm{mod}\,P}, l_{\mathrm{mod}\,Q})||| \end{array} \right) < 1.$

(14)

Proof: The proof is omitted for brevity and is given in [10].

It can be shown [10] that the conditions given in Theorem 2 are more restrictive than those in Theorem 1. However the computation required is less in Theorem 2 than in

Theorem 1. Therefore, for a given 2-D PSV system, we first use Theorem 2 to check the stability. If it fails, we then use Theorem 1. Now we present the necessary conditions.

Theorem 3 *Consider the zero-input 2-D PSV state space model given in* (2)*. Define* $U_l = [I - T_l]^{-1} S_l$*, where*

$$T_l = \begin{bmatrix} \bar{0} & \cdots & & A_1(P\text{-}1,l) \\ A_1(0,l) & \ddots & & \vdots \\ \vdots & \ddots & & \\ \bar{0} & & A_1(P\text{-}2,l) & \bar{0} \end{bmatrix},$$

$$S_l = \begin{bmatrix} A_2(P\text{-}1,l) & \bar{0} & \cdots & A_0(P\text{-}1,l) \\ A_0(0,l) & A_2(0,l) & & \vdots \\ \vdots & \ddots & \ddots & \\ \bar{0} & & A_0(P\text{-}2,l) & A_2(P\text{-}2,l) \end{bmatrix}$$

and $\bar{0}$ *denotes the all-zero matrix of appropriate dimension. If the system is stable, i.e.,* $lim_{i+j\to\infty}\, x(i,j) = 0$*, then*

$$\rho\left(\prod_{l=Q-1}^{0} U_l\right) < 1. \tag{15}$$

where ρ *denotes the spectral radius of a matrix.*

The above theorem is proved in [10] using the fact that since the system is stable, the energy on the horizontal lines diminish to zero. If we use the same idea for the vertical lines, then we get another necessary condition which is given in the next theorem. The proofs of Theorems 3 and 4 are also omitted and can be found in [10].

Theorem 4 *Consider the zero-input 2-D PSV state space model given in* (2)*. Define* $Z_l = [I - X_l]^{-1} Y_l$*, where*

$$X_l = \begin{bmatrix} \bar{0} & \cdots & & A_2(l,Q\text{-}1) \\ A_2(l,0) & \ddots & & \vdots \\ \vdots & \ddots & & \\ \bar{0} & & A_2(l,Q\text{-}2) & \bar{0} \end{bmatrix},$$

$$Y_l = \begin{bmatrix} A_1(l,Q\text{-}1) & \bar{0} & \cdots & A_0(l,Q\text{-}1) \\ A_0(l,0) & A_1(l,0) & & \vdots \\ \vdots & \ddots & \ddots & \\ \bar{0} & & A_0(l,Q\text{-}2) & A_1(l,Q\text{-}2) \end{bmatrix}$$

and $\bar{0}$ *denotes the all-zero matrix of appropriate dimension. If the system is stable, i.e.,* $lim_{i+j\to\infty}\, x(i,j) = 0$*, then*

$$\rho\left(\prod_{l=P-1}^{0} Z_l\right) < 1. \tag{16}$$

where ρ *denotes the spectral radius of a matrix.*

Due to space limitations, examples are omitted. Several examples illustrating the theorems can be found in [10].

5. CONCLUSION

The stability of 2-D PSV systems represented as the *first* model of Fornasini-Marchesini has been studied in this paper. Four theorems are established for checking the stability of the considered system. Two of them are sufficient conditions and the others are necessary conditions. Among the sufficient conditions, Theorem 2 is more restrictive than Theorem 1 but it is easier and faster to apply.

6. REFERENCES

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