

# THE DESIGN OF HILBERT TRANSFORM PAIRS OF WAVELET BASES VIA THE FLAT DELAY FILTER

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## ABSTRACT

This paper describes a simple procedure, based on spectral factorization, for the design of a pair of orthonormal wavelet bases where the two wavelets form a Hilbert transform pair. The two scaling filters respectively have the numerator and denominator of the flat delay all-pass filter as factors. The design procedure allows for an arbitrary number of zero wavelet moments to be specified. A Matlab program for the procedure is given, and examples are also given to illustrate the results.

## 1. INTRODUCTION

This paper describes a simple procedure for the construction of two orthogonal wavelet bases designed so that the two wavelets form a Hilbert transform pair. Several authors have advocated the simultaneous use of two wavelet transforms where the wavelets are so related. For example, Freeman and Adelson employ the Hilbert transform in the development of steerable filters [1], Abry suggested it for the analysis of turbulence [2], Ozturk, Kucur, and Atkin suggested it for waveform encoding [3], and Kingsbury developed the complex dual-tree DWT [4, 5] and illustrated its numerous benefits including near shift-invariance and improved denoising capability.

Using the infinite product formula, it was shown in [6] that for two orthogonal wavelets to form a Hilbert transform pair, the scaling filters should be offset by a half sample. In [6] a design problem was formulated for the minimal length scaling filters such that (i) the wavelets each have a specified number of zero moments ( $K$ ), and (ii) the half-sample delay approximation is flat at  $\omega = 0$  with specified degree ( $L$ ). However, this formulation leads to nonlinear design equations, and the examples in [6] had to be obtained using Gröbner bases. In the paper we describe a design procedure based on spectral factorization. It results in filters similar to those of [6], however the design algorithm is much simpler.

### 1.1. Preliminaries

Let the filters  $h_0(n)$ ,  $h_1(n)$  represent a CQF pair [7]. That is,

$$\sum_n h_0(n) h_0(n+2k) = \delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0, \end{cases}$$

and  $h_1(n) = (-1)^{(1-n)} h_0(1-n)$ . Equivalently, in terms of the  $Z$ -transform, we have

$$H_0(z)H_0(1/z) + H_0(-z)H_0(-1/z) = 2$$

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and

$$H_1(z) = \frac{1}{z} H_0(-1/z).$$

Let the filters  $g_0(n)$ ,  $g_1(n)$  represent a second CQF pair. In this paper we assume  $h_i(n)$ ,  $g_i(n)$  are real-valued filters. It is convenient to write the CQF condition in terms of the autocorrelation functions, defined as

$$p_h(n) := h_0(n) * h_0(-n), \quad p_g(n) := g_0(n) * g_0(-n),$$

or equivalently as

$$P_h(z) := H_0(z) H_0(1/z), \\ P_g(z) := G_0(z) G_0(1/z).$$

Then  $h_0(n)$  and  $g_0(n)$  satisfy the CQF conditions if and only if  $p_h(n)$  and  $p_g(n)$  are halfband filters:

$$p_h(n) = \begin{cases} 1 & n = 0 \\ 0 & n = \pm 2, \pm 4, \dots \end{cases}$$

and similarly for  $p_g(n)$ . This can be written more compactly as

$$p_h(2n) = \delta(n) \quad \text{and} \quad p_g(2n) = \delta(n). \quad (1)$$

The dilation and wavelet equations give the scaling and wavelet functions,

$$\phi_h(t) = \sqrt{2} \sum_n h_0(n) \phi_h(2t - n) \\ \psi_h(t) = \sqrt{2} \sum_n h_1(n) \phi_h(2t - n).$$

The scaling function  $\phi_g(t)$  and wavelet  $\psi_g(t)$  are defined similarly, but with filters  $g_0(n)$  and  $g_1(n)$ .

Notation: The  $Z$ -transform of  $h(n)$  is denoted by  $H(z)$ . The discrete-time Fourier transform of  $h(n)$  will be denoted by  $H(\omega)$ , although it is an abuse of notation. The Fourier transform of  $\psi(t)$  is denoted by  $\Psi(\omega) = \mathcal{F}\{\psi(t)\}$ .

### 1.2. Hilbert transform pairs

In [6], it was shown that if  $H_0(\omega)$  and  $G_0(\omega)$  are lowpass CQF filters with

$$G_0(\omega) = H_0(\omega) e^{-j\frac{\omega}{2}} \quad \text{for} \quad |\omega| < \pi,$$

then the corresponding wavelets are a Hilbert transform pair,

$$\psi_g(t) = \mathcal{H}\{\psi_h(t)\}.$$

That is,

$$\Psi_g(\omega) = \begin{cases} -j \Psi_h(\omega), & \omega > 0 \\ j \Psi_h(\omega), & \omega < 0. \end{cases}$$

Equivalently, the digital filter  $g_0(n)$  is a *half-sample* delayed version of  $h_0(n)$ ,

$$g_0(n) = h_0(n - 1/2).$$

As a half-sample delay can not be implemented with an FIR filter (not even a rational IIR filter can be exact), it is necessary to make an approximation.

## 2. DESIGN PROCEDURE

In this paper, we look for solutions of the following form

$$\begin{aligned} h_0(n) &= f(n) * d(n), \\ g_0(n) &= f(n) * d(L - n), \end{aligned}$$

where the filter  $d(n)$  will be chosen to achieve the (approximate) half-sample delay. The first step of the design procedure will be to determine the appropriate filter  $d(n)$  so as to achieve the desired relationship between  $h_0(n)$  and  $h_1(n)$ . In terms of the transfer functions, we have

$$\begin{aligned} H_0(z) &= F(z) D(z) \\ G_0(z) &= F(z) z^{-L} D(1/z). \end{aligned}$$

$H_0(z)$  and  $G_0(z)$  have the common divisor  $F(z)$  which will be determined later. We can write

$$G_0(z) = H_0(z) \frac{z^{-L} D(1/z)}{D(z)}$$

where we can recognize that the transfer function

$$A(z) := \frac{z^{-L} D(1/z)}{D(z)}$$

is an all-pass system,  $|A(\omega)| = 1$ . Therefore

$$|G_0(\omega)| = |H_0(\omega)|, \quad |G_1(\omega)| = |H_1(\omega)|$$

and

$$|\Psi_g(\omega)| = |\Psi_h(\omega)|.$$

If the all-pass system  $A(z)$  is an approximate half-sample delay,

$$A(\omega) \approx e^{-j\omega/2}$$

or equivalently,  $A(z) \approx z^{-1/2}$ , then the sought approximation is achieved,

$$G_0(\omega) \approx H_0(\omega) e^{-j\frac{\omega}{2}}.$$

Several authors have addressed the design of  $D(z)$  such that the all-pass system  $A(z)$  approximates a fractional delay [8, 9, 10]. The following formula for the maximally flat delay all-pass filter

is adapted from Thiran's formula for the maximally flat delay all-pole filter [11]. The maximally flat approximation to a delay of  $\tau$  samples is given by

$$D(z) = 1 + \sum_{n=1}^L d(n) z^{-n}$$

where

$$d(n) = (-1)^n \binom{L}{n} \frac{(\tau - L)_n}{(\tau + 1)_n} \quad (2)$$

and  $(x)_n$  represents the rising factorial,

$$(x)_n := \underbrace{(x)(x+1) \cdots (x+n-1)}_{n \text{ terms}}.$$

With this  $D(z)$  we have the approximation

$$A(z) = \frac{z^{-L} D(1/z)}{D(z)} \approx z^{-\tau} \quad \text{around } z = 1$$

and

$$G_0(\omega) = H_0(\omega) e^{-j\tau\omega} \quad \text{around } \omega = 0.$$

In our problem, we will use  $d(n)$  in (2) with  $\tau = 1/2$ . They can be computed very efficiently using the following ratio.

$$\begin{aligned} \frac{d(n+1)}{d(n)} &= -\frac{\binom{L}{n+1}}{\binom{L}{n}} \cdot \frac{(\tau - L)_{n+1}}{(\tau - L)_n} \cdot \frac{(\tau + 1)_n}{(\tau + 1)_{n+1}} \\ &= \frac{(L - n)(L - n - \tau)}{(n + 1)(n + 1 + \tau)} \end{aligned}$$

From this ratio, it follows that the filter  $d(n)$  can be generated as follows.

$$d(0) = 1$$

$$d(n+1) = d(n) \cdot \frac{(L - n)(L - n - \tau)}{(n + 1)(n + 1 + \tau)}, \quad 0 \leq n \leq L - 1.$$

This can be implemented in Matlab with only two commands.

```
n = 0:L-1;
d = cumprod([1, (L-n).*(L-n-t)./(n+1)./(n+1+t)]);
```

To obtain wavelet bases with  $K$  zero moments, we let

$$F(z) = Q(z) (1 + z^{-1})^K.$$

Then

$$H_0(z) = Q(z) (1 + z^{-1})^K D(z), \quad (3)$$

$$G_0(z) = Q(z) (1 + z^{-1})^K z^{-L} D(1/z). \quad (4)$$

We now have the following design problem. Given  $D(z)$  and  $K$ , find  $Q(z)$  of minimal degree such that  $h_0(n)$  and  $g_0(n)$  satisfy the CQF conditions (1). Note that with (3, 4)  $h_0(n)$  and  $g_0(n)$  have the same autocorrelation function:

$$\begin{aligned} P(z) &:= P_h(z) = P_g(z) \\ &= Q(z) Q(1/z) (z + 2 + z^{-1})^K D(z) D(1/z). \end{aligned}$$

Similar to the way Daubechies wavelet filters are obtained, we can obtain  $Q(z)$  using a spectral factorization approach as in [7]. The procedure consists of two steps.

1. Find  $r(n)$  of minimal length such that

- (a)  $r(n) = r(-n)$
- (b)  $R(z) (z + 2 + z^{-1})^K D(z) D(1/z)$  is halfband.

Note that  $r(n)$  of minimal length will be supported on the range  $(1 - K - L) \leq n \leq (K + L - 1)$ .

2. Set  $Q(z)$  to be a spectral factor of  $R(z)$ ,

$$R(z) = Q(z) Q(1/z). \quad (5)$$

To carry out the first step, we need only solve a system of linear equations. Defining

$$S(z) := (z + 2 + z^{-1})^K D(z) D(1/z)$$

we can write the halfband condition as

$$\begin{aligned} \delta(n) &= [\downarrow 2] (s * r)(n) \\ &= \sum_k s(2n - k) r(k). \end{aligned}$$

When written in matrix form, this calls for a square matrix of dimension  $2(K + L) - 1$  which has the form of a convolution (Toeplitz) matrix with every second row deleted.

The second step assumes  $R(z)$  permits spectral factorization, which we have found to be true in all our examples. With  $Q(z)$  obtained in this way, the filters  $H_0(z)$  and  $G_0(z)$  defined in (3, 4) satisfy the CQF conditions and have the desired half-sample delay. Note that  $Q(z)$  is not unique.

This design procedure yields filters  $h_0(n)$  and  $g_0(n)$  of (minimal) length  $2L + 2K$ . A Matlab program to implement this design procedure is given Table 1. The commands `binom` and `sfact` for computing binomial coefficients and performing spectral factorization are not currently standard Matlab commands. They are available from the author.

## 2.1. Examples

**Example 1:** With  $K = 4$  and  $L = 2$  the filters  $h_0(n)$  and  $g_0(n)$  are of length 12. Fig. 1 illustrates a solution obtained from a mid-phase spectral factorization.  $|H_0(\omega)|$  and  $|G_0(\omega)|$  are identical. The plot of the phase of  $G_0(\omega)/H_0(\omega)$ , denoted by  $\theta(\omega)$ , shows its agreement with  $\omega/2$  near  $\omega = 0$ . The plot of the function  $|\Psi_h(\omega) + j \Psi_g(\omega)|$  shows that it approximates zero for  $\omega < 0$  as expected if  $\psi_h$  and  $\psi_g$  make a Hilbert transform pair.

**Example 2:** With  $K = 3$  and  $L = 3$  the filters  $h_0(n)$  and  $g_0(n)$  are again of length 12. Fig. 2 illustrates a solution using a mid-phase spectral factor. It can be seen that  $|\Psi_h(\omega) + j \Psi_g(\omega)|$  is closer to zero for negative frequencies. This is to be expected, as we have reduced the number of zero moments and at the same time increased the degree of approximation for the half-sample delay.

## 3. CONCLUSION

This paper describes a simple procedure, based on spectral factorization, for the design of a pair of orthonormal wavelet bases where the two wavelets form a Hilbert transform pair. Given the flat delay all-pass filter and a specified number of zero wavelet derivatives, the procedure as outlined in this paper yields the CQF filters of minimal length. (However, any other all-pass filter that approximates a delay of a half sample could also be used.) A Matlab program for the procedure is given, and examples are also given to illustrate the results.

**Table 1.** Matlab program.

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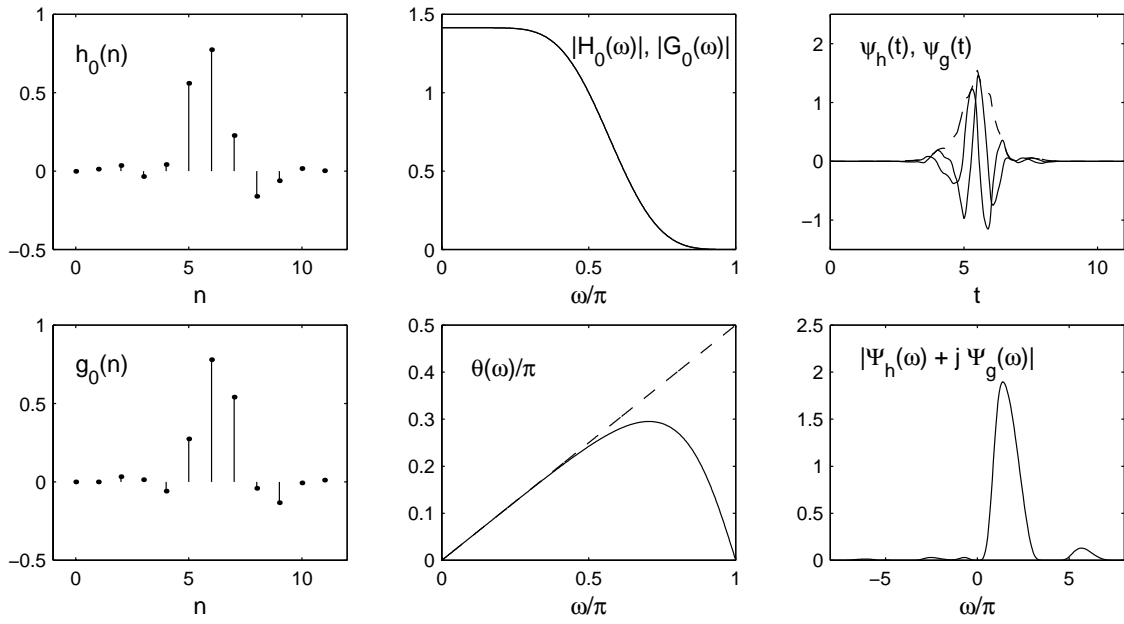
```
function [h,g] = hwlet(K,L)
% Hilbert transform pair of orthogonal wavelet bases
% h, g - scaling filters of length 2*(K+L)
% K - number of zeros at z=-1
% L - degree of fractional delay

n = 0:L-1;
t = 1/2;
d = cumprod([1, (L-n).*(L-n-t)./(n+1)./(n+1+t)]);
s1 = binom(2*K,0:2*K);
s2 = conv(d,d(end:-1:1));
s = conv(s1,s2);
M = K+L;
C = convmtx(s',2*M-1);
C = C(2:2:end,:);
b = zeros(2*M-1,1);
b(M) = 1;
r = (C\b)';
q = sfact(r);
f = conv(q,binom(K,0:K));
h = conv(f,d);
g = conv(f,d(end:-1:1));
```

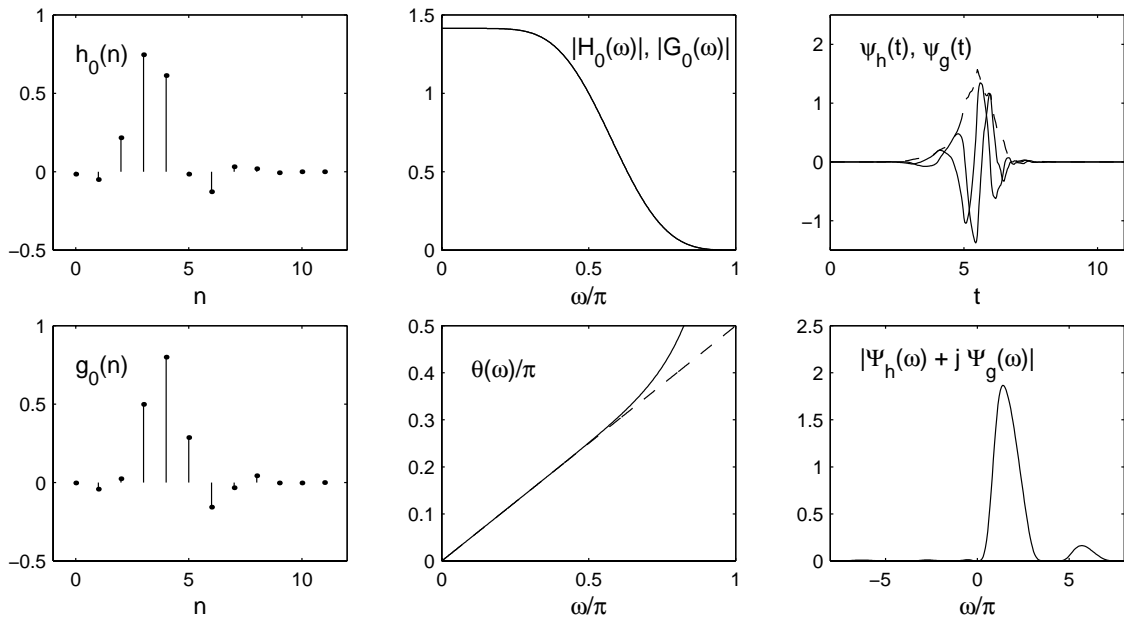
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## 4. REFERENCES

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**Fig. 1.** Example of near Hilbert transform pair of orthonormal wavelet bases, with  $N = 12$ ,  $K = 4$ ,  $L = 2$ .



**Fig. 2.** Example of near Hilbert transform pair of orthonormal wavelet bases, with  $N = 12$ ,  $K = 3$ ,  $L = 3$ .