AN ITERATIVE SOLUTION FOR THE OPTIMAL POLES IN A KAUTZ SERIES

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ABSTRACT

Kautz series allow orthogonal series expansion of finite-energy signals defined on a semi-infinite axis. The Kautz series consists of orthogonalized exponential functions or sequences. This series has as free parameters an ordered set of poles, each pole associated with an exponential function or sequence. For reasons of approximation and compact representation (coding), an appropriate set of ordered poles is therefore convenient. An iterative procedure to establish the optimal parameters according to an enforced convergence criterion is introduced.

1. INTRODUCTION

In the time-domain, processes are linked to causal operators. For practical purposes, causal operators characterized by linearity, stability and finite number of states form the most prominent class of systems. These are associated with rational transfer functions and, in the time domain, with damped exponential functions or sequences.

Infinite series of (generalized) exponential functions or sequences constitute under some conditions a complete basis in $L_2(0,\infty)$ or $\ell_2(\mathbb{N}_0)$ (next section). From this, an orthonormal basis can be constructed which is usually called a Kautz basis [1, 2]. For approximation and compact representation, fast converging series are of interest. The poles in the series can be adapted to achieve this.

This line of work has been topic of recent research. Necessary conditions for the optimal poles according to a squared error criterion in a truncated series expansions have been derived for Laguerre series [3, 4, 5], Kautz series with two repeated poles [6] and sets of repeated poles [7]. As a result, we have optimality conditions but a search procedure is still necessary to find the poles [8]. More recently, these optimality conditions have been extended to L^p norms and linearly constrained expansion coefficients [9].

Another approach has been to formulate an enforced convergence which defines the optimal poles. Using an appropriate criterion, optimal poles can be established based on some signal measurements only and being independent of the number of terms in the truncation. We mention the work on optimal parameters in a Laguerre series expansion [10, 5] and harmonically modulated Laguerre series [11]. The basis for these results is their connection to orthogonal polynomials and the properties thereof, in particular the differential or difference equation [12]. In this paper, a similar procedure as for the Laguerre series is used but now iteratively to select one-by-one the different poles in a general Kautz series.

The outline of this paper is as follows. We start with the definition of the Kautz system. For convenience, we consider discrete-time functions only, though the results carry over to the continuous time. Next, a difference equation is derived and an enforced convergence criterion is formulated for a specific Kautz system, i.e., governed by a fixed set of poles and an extra free pole determining the basis in the remaining subspace. This pole can be optimized according to the criterion. As a consequence we can formulate an iterative solution. An example and a discussion ends the paper.

2. KAUTZ SERIES

We denote the n-th Kautz functions by $\phi_n(t,\theta)$ where $t \in \mathbb{N}_0$ and θ is a parameter set which will be explained later. As a shorthand notation we use $\phi_n[\theta]$ as well. We introduce these function by their z-transforms.

The z-transform $\Phi_n[\theta]$ of the n-th Kautz function $\phi_n[\theta]$ is given by:

$$\Phi_n(z;\theta) = \kappa_n\sqrt{1-p_np_n^*}\,\frac{z}{z-p_n}\prod_{l=0}^{n-1}\frac{1-zp_l^*}{z-p_l} \qquad (1)$$

with $|\kappa_n| = 1$, $\theta = (p_0, p_1, p_2, \ldots)$, $|p_n| < 1$. If all poles are identical, $p_n = \rho$, and real-valued, we have the Laguerre basis, in this case we use $\theta = \rho(1,1,\ldots)$:

$$\Phi_n(z;\theta) = \sqrt{1-\rho^2}\,\frac{z}{z-\rho}\left(\frac{1-z\rho}{z-\rho}\right)^n.$$

The functions $\phi_n[\theta]$ are now real-valued. If all poles are identical $p_n = \rho$ with $\Im\{\rho\} \neq 0$, we will call the corresponding orthonormal system the complex Laguerre basis:

$$\Phi_n(z;\theta) = \sqrt{1-\rho\rho^*}\,\frac{z}{z-\rho}\left(\frac{1-z\rho^*}{z-\rho}\right)^n.$$

The system $\{\phi_n[\theta] | n \in \mathbb{N}_0\}$ is a complete orthonormal system in $\ell_2(\mathbb{N}_0)$ if and only if $\sum_n (1 - |p_n|)$ diverges [13].

We introduce the l-fixed pole Kautz series as Kautz series where the first l poles are fixed and $p_n = p$ for $n \geq l$; with p a free parameter. In terms of $\theta = \theta_l(p)$:

$$\theta_l(p) = (p_0, p_1, p_2, \dots, p_{l-1}, p, p, \dots).$$

The l-fixed pole Kautz series adheres to a single difference equation. This difference equation formulated in the z-domain reads as follows. For $n \geq l$ the n-th Kautz function satisfies:

$$\left[\tfrac{-pz^{-1} + pp^*}{1 - pp^*} + l - n - \tfrac{(z-p)(1-p^*z)}{1-pp^*} h_l(z)\right] \Phi_n[\theta_l] - \tfrac{(z-p)(1-p^*z)}{1-pp^*} \Phi_n'[\theta_l] = 0, \quad (2)$$

where

$$h_l(z) := \sum_{k=0}^{l-1} \frac{1 - p_k p_k^*}{(z - p_k)(1 - p_k^* z)}.$$

The admissible space $\mathcal{D}$ for the Kautz series expansions is defined by

$$\mathcal{D} := \{x | x, \sqrt{t}x \in \ell_2(\mathbb{N}_0)\}.$$

The space $\mathcal{D}$ is a Hilbert subspace of $\ell_2(\mathbb{N}_0)$ of sequences having finite energy and finite first-order moment (center of the energy). For all practical purposes, the restrictions on x are very mild. For convenience, we restrict ourselves in the next sections to real-valued poles. For an extension to the general case of complex poles, we refer to [14].

3. OPTIMALITY CRITERION

In the time-domain the l-fixed pole Kautz differential equation corresponds to a finite-order difference equation. Since the z-transformation $\mathcal{Z}$ maps unitarily $\ell_2(\mathbb{N}_0)$ onto $\mathcal{H}_2(\mathbb{E})$, and since the expressions in terms of the differential equation (2) are simpler in the z-domain than in the time-domain, we define the enforced convergence criterion $\mathcal{Q}$ in the z-domain. With $X = \mathcal{Z}x$, we define

$$\mathcal{Q}(x; \theta_l(p)) := \sum_{n=0}^{\infty} n |\langle X, \Phi_n[\theta_l(p)] \rangle|^2.$$

To determine the optimal pole in a l-fixed Kautz series we come to the following problem.

Minimization Problem: *Given $X = \mathcal{Z}x$, determine p minimizing*

$$\mathcal{Q}(x; \theta_l(p)) = \sum_{n=0}^{\infty} n |\langle X, \Phi_n[\theta_l(p)] \rangle|^2.$$

□

Using the properties of the l-fixed pole Kautz difference equation (2), it can be proved that there exist functionals $c_{l1}(X)$, $c_{l2}(X)$ and $c_{l3}(X)$ such that [14]

$$\mathcal{Q}(x; \theta_l) = c_{l1}(X) \frac{p}{1 - p^2} + c_{l2}(X) \frac{1}{1 - p^2} + c_{l3}(X). \quad (3)$$

In the next section we give the explicit expression of the functionals $c_{l1}(X)$, $c_{l2}(X)$ and $c_{l3}(X)$. The optimal parameter in $(-1, 1)$ can now be calculated. To avoid problems for the trivial function $x = 0$, the space $\mathcal{D}^*$ is defined as the space $\mathcal{D}$ with the exclusion of the function $x = 0$.

Lemma: *[14] In the Kautz series expansions, the optimum parameter for the enforced convergence criterion $\mathcal{Q}(x; \theta_l)$, where the first l p_k's are taken fixed, is given by*

$$\hat{p} = \frac{c_{l2}(X)}{c_{l1}(X)} \left(\sqrt{1 - \frac{c_{l1}^2(X)}{c_{l2}^2(X)}} - 1\right), \quad (4)$$

where $x \in \mathcal{D}^*$. The minimum enforced convergence criterion is

$$\mathcal{Q}(x; \hat{\theta}_l) = -\frac{1}{2} c_{l1}(X) \sqrt{1 - \frac{c_{l1}^2(X)}{c_{l2}^2(X)}} \left[1 + \sqrt{1 - \frac{c_{l1}^2(X)}{c_{l2}^2(X)}}\right] + c_{l3}(X).$$

where $\hat{\theta}_l = (p_0, p_1, \dots, p_{l-1}, \hat{p}, \hat{p}, \dots)$. □

It can be shown that with each new pole calculated according to the previous procedure, the optimality criterion reduces. More formally [14], we have:

Theorem: *[14] Let x be a function in the admissible space $\mathcal{D}$ which is not equal to zero, and*

$$\theta_l(p) = (p_0, p_1, \dots, p_{l-1}, p, p, \dots),$$

where $p, p_k \in (-1, 1)$, $k = 0, 1, \dots, l-1$, $l \in \mathbb{N}$, and p_k's are fixed. Further, let $\theta_{l+1}(p) = (p_0, p_1, \dots, p_l, p, p, \dots)$, where $p_l = \hat{p}$ is the optimal solution of

$$\min_{p \in [-1,1]} \mathcal{Q}(x; \theta_l(p)).$$

Then

$$\min_{p \in [-1,1]} \mathcal{Q}(x; \theta_{l+1}) \leq \min_{p \in [-1,1]} \mathcal{Q}(x; \theta_l).$$

□

4. ALGORITHM

We recall the enforced convergence criterion:

$$\mathcal{Q}(x; \theta_l) = c_{l1}(X) \frac{p}{1 - p^2} + c_{l2}(X) \frac{1}{1 - p^2} + c_{l3}(X),$$

where $c_{l1}(X), c_{l2}(X)$ and $c_{l3}(X)$ are independent of p and defined [14] by

$$\begin{aligned} c_{l1}(X) =& b_{l1}(X,X) - \sum_{n=0}^{l-1} \langle X, \Phi_n[\theta_l] \rangle b_{l1}(\Phi_n[\theta_l], X), \\ c_{l2}(X) =& b_{l2}(X,X) - \sum_{n=0}^{l-1} \langle X, \Phi_n[\theta_l] \rangle b_{l2}(\Phi_n[\theta_l], X), \\ c_{l3}(X) =& b_{l3}(X,X) + \sum_{n=0}^{l-1} \langle X, \Phi_n[\theta_l] \rangle \Big[n \langle \Phi_n[\theta_l], X \rangle \\ & - b_{l3}(\Phi_n[\theta_l], X) \Big], \end{aligned}$$

with

$$\begin{aligned} b_{l1}(X,Y) =& \langle X', (1+z^{-2})Y \rangle - \langle z^{-1}X, Y \rangle + \\ & + \sum_{k=0}^{l-1} (1-p_k^2) \Big[\langle \frac{1}{z-p_k} X, \frac{z}{z-p_k} Y \rangle \\ & + \langle \frac{z}{z-p_k} X, \frac{1}{z-p_k} Y \rangle \Big], \\ b_{l2}(X,Y) =& -2 \langle X', z^{-1}Y \rangle + \langle X, Y \rangle \\ & - 2 \sum_{k=0}^{l-1} (1-p_k^2) \langle \frac{1}{z-p_k} X, \frac{1}{z-p_k} Y \rangle, \\ b_{l3}(X,Y) =& \langle X', z^{-1}Y \rangle + (l-1) \langle X, Y \rangle \\ & + \sum_{k=0}^{l-1} (1-p_k^2) \langle \frac{1}{z-p_k} X, \frac{1}{z-p_k} Y \rangle. \end{aligned}$$

For the recursive solution of the real Kautz system we have the following algorithm:
Algorithm: For $x \in \mathcal{D}^*$:

1. $N \in \mathbb{N}$;
2. do $l = 0 : N-1$,
   $\beta := \frac{c_{l2}(X)}{c_{l1}(X)}$;
   $p_l := \beta \left(\sqrt{1 - \frac{1}{\beta^2}} - 1 \right)$;
   od;

□

5. EXAMPLE

As an example, we took the function x defined by a rational z-transform:

$$X(z) = \frac{\prod_{i=1}^{3} (z - q(i))}{\prod_{i=1}^{4} (z - p(i))},$$

with

$$\begin{aligned} q &= (-1, 1, 1), \\ p &= (0.3, -0.3, 0.4, -0.4). \end{aligned}$$

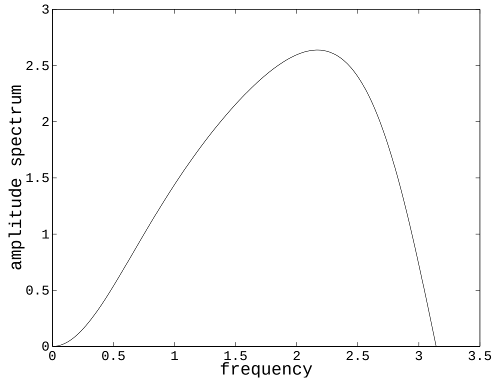

Fig. 1. Amplitude spectrum of the function x

The function x is real-valued. In Fig. 1 the amplitude spectrum is given. We observe a band-pass filter. The optimal Laguerre solution [10, 5] is given by $\hat{\rho} = -0.1833$. Applying the algorithm described in the previous section to this function for $N = 6$ we obtain for the optimal first 6 parameters:

$$\begin{aligned} \hat{\theta}^{1-6} = (& -0.1833, -0.1567, -0.2728, 0.4526, 0.1091, \\ & -0.262). \end{aligned}$$

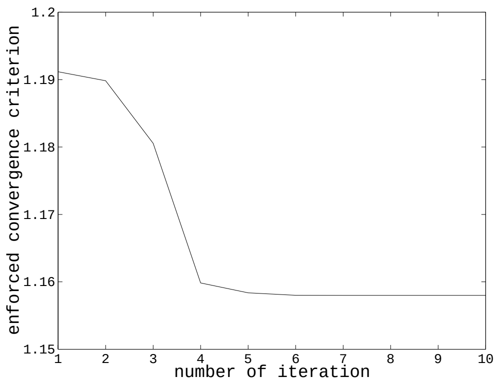

Fig. 2. Enforced convergence criterion as function of the iteration

In Fig. 2, the relative value of $\mathcal{Q}(x;\theta)$ $(= \mathcal{Q}(x;\theta)/\|x\|^2)$ is shown as a function of the number of iterations. The relative enforced criterion is decreasing as function of the number of iterations. We see that $\mathcal{Q}(x;\theta)$ becomes quite constant after 6 iterations.

We now compare the results of the proposed procedure to that of using a Laguerre series with optimized poles. As a measure we consider the relative loss in energy using a truncated series expansion. The relative loss is defined as

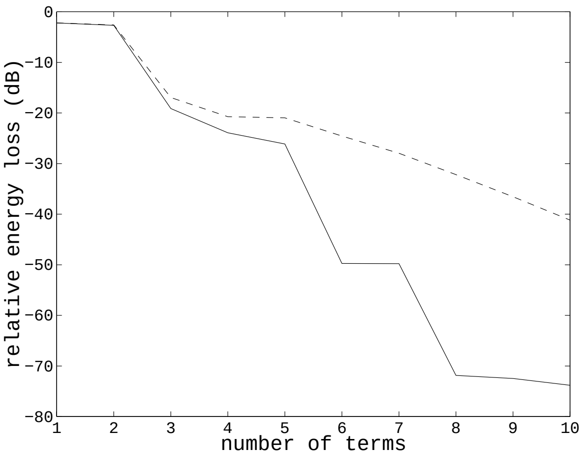

Fig. 3. Relative energy loss in the Kautz series with one optimal pole (dashed line) and recursively with different poles (solid line)

$(E - E_N)/E$ where E is the energy in the original function x and and E_N the energy in the approximation obtained by truncating the series expansion at the N-th term.

In Fig. 3, the relative energy loss of x is shown as a function of the number of sections in a truncated Kautz series (solid line) where the poles were optimized by the proposed algorithm. We also calculated the relative loss of energy for a truncated Laguerre series where the pole was optimized ($\hat{p} = -0.1833$) according to the enforced convergence criterion for the discrete Laguerre case (dashed line). We see that the Kautz series is better than the Laguerre series in this case. We note that this example is representative for functions taken in span$\{\lambda_0^t, \lambda_1^t, \ldots, \lambda_m^t\}$.

6. DISCUSSION

We have considered the case of Kautz series and the selection of the optimal poles according to an enforced convergence criterion. The work is related to earlier work on optimal poles for Laguerre series and can be viewed as an adapted, repeated application of calculating the optimal poles in a Laguerre series. By an algorithm, we have shown how to determine the optimal parameters in the recursive scheme. By an example, we have illustrated the behaviour of the algorithm and compared the result to those using a Laguerre series.

The proposed algorithm can be extended to the case of complex poles by appropriate adaptation of the definition of $\mathcal{Q}$, the b's and c's, see [14]. The optimality criterion can be adapted to yield faster convergence but, in general, only at the cost of more elaborate algorithms [14].

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