ASYMPTOTIC PERFORMANCE OF ML CHANNEL ESTIMATORS IN WCDMA SYSTEMS: RANDOMIZED CODES APPROACH

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ABSTRACT

This paper analyzes the asymptotic performance of Maximum Likelihood (ML) channel estimation algorithms in wideband code division multiple access (WCDMA) scenarios. We concentrate on systems with periodic spreading sequences (period larger than or equal to the symbol span) with high spreading factors, where the transmitted signal contains a code division multiplexed pilot for channel estimation purposes. Assuming randomized training and code sequences, we derive and compare the asymptotic covariances of the training-only (TO), semi-blind conditional ML (CML) and semi-blind Gaussian ML (GML) channel estimators.

1. INTRODUCTION

Multi-rate code division multiple access systems –such as WCDMA– have recently been proposed for third generation terrestrial and satellite mobile communication applications. One of the most interesting features of these systems is the introduction of a pilot signal which, serving channel estimation purposes, is code division multiplexed and transmitted at the same time as the traffic information. Given the importance that such systems may have in the near future, it seems crucial to answer questions such as what is the dependence of the channel estimation performance on e.g. the spreading codes repetition period, the signal to noise ratio, the quotient between the pilot and the traffic signal powers or the spreading factor employed. This paper tries to provide answers to these questions in a theoretical yet simple manner. It will be shown that under some asymptotic conditions very simple expressions describing the mean behavior of the analyzed channel estimation algorithms can be obtained. These simple expressions will prove most useful from a system designer point of view in that they will provide a precise description of the expected performance as a function of physical system parameters.

From the whole range of channel estimation algorithms available in the literature, we will focus on Maximum Likelihood (ML) estimation procedures, since under certain regularity conditions they provide asymptotically efficient estimates. In particular, we will study both classical training-only techniques (based on the knowledge of the pilot or training sequence exclusively) and semi-blind techniques (which make use of the training sequence and improve this estimation taking into account the signal structure).

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2. MULTI-RATE CDMA SIGNAL MODEL

The transmitting station is assumed to map the underlying data sequence to Q distinct and synchronized spreading sequences with period N_c chips. Both the period of the spreading sequences and the chip rate are assumed constant for all the sequences. Let $s_q(m) \in \mathbb{C}$ represent the underlying complex symbol stream associated with the qth code sequence and assume that $N_s(q),\ q = 1 \ldots Q$ consecutive symbols are mapped to each interval of N_c chips. The spreading factor associated with a particular spreading sequence is denoted as $SF_q = N_c/N_s(q)$ chips/bit. According to these definitions, the total number of symbols mapped to a code sequence period (N_c chips) is $N_s = \sum_{q=1}^{Q} N_s(q)$, where it is assumed that $N_s < N_c$.

We assume that at the basestation the signal is synchronously sampled at the chip rate (modulation with no excess bandwidth) and that symbol detection is made in observation windows of MN_c chips, where M can be regarded as the number of spreading periods in the observation interval. Stacking MN_c samples of the received signal into a column vector $\mathbf{x} \in \mathbb{C}^{MN_c \times 1}$ we can describe the received signal as

$$\mathbf{x} = \mathbf{x}^k + \mathbf{G}\mathbf{s} + \mathbf{n}, \tag{1}$$

with $\mathbf{x}^k$ the channel-filtered training sequence, $\mathbf{G}$ a matrix of received signatures, $\mathbf{s}$ a vector containing the complex-valued transmitted symbols and $\mathbf{n}$ the noise component. Note that the training sequence is transmitted code division multiplexed (rather than time division multiplexed) with the traffic information. The column vector $\mathbf{h} = \left[h(1) \cdots h(L)\right]^T \in \mathbb{C}^{L \times 1}$ contains the channel impulse response, assumed constant within the observation interval and of length $L < N_c$. Accordingly, the known part of the received signal $\mathbf{x}^k$ can be expressed as

$$\mathbf{x}^k = \mathcal{T}\mathbf{h}, \quad \mathcal{T} = \begin{bmatrix} t(1) & \cdots & t(-L+2) \\ \vdots & \ddots & \vdots \\ t(MN_c) & \cdots & t(MN_c - L + 1) \end{bmatrix}, \tag{2}$$

where $t(n)$ denotes the chip-level complex-valued training sequence.

Let us now concentrate on the matrix of received signatures $\mathbf{G}$. This matrix is formed stacking side by side the signature matrices corresponding to each of the code sequences $\mathbf{G} = [\mathbf{G}_1 \cdots \mathbf{G}_Q]$. These $\mathbf{G}_q$, $q = 1 \ldots Q$, can in turn be expressed as

$$\mathbf{G}_q = \mathbf{C}_q \left(\mathbf{I}_{M_s(q)} \otimes \mathbf{h}\right)$$
$$\mathbf{C}_q = \begin{bmatrix} \mathcal{D}_q(2) \, \mathcal{C}_q(1) & \cdots & \mathbf{0} \\ \mathbf{0} \quad \mathcal{C}_q(2) & \ddots & \vdots \\ \vdots \quad\quad \ddots & \ddots & \mathbf{0} \\ \mathbf{0} \quad\quad \cdots & \mathcal{C}_q(2) & \mathcal{C}_q(1) \end{bmatrix}_{M \times M+1 \text{ blocks}}, \quad (3)$$

with $\otimes$ the Kronecker product and $M_s(q) = \left\lceil \frac{L-1}{SF_q} \right\rceil + MN_s(q)$ the effective number of symbols received in an observation interval. Matrices $\mathcal{C}_q(1)$ and $\mathcal{C}_q(2)$ are the upper and lower parts of the next convolution matrix:

$$\begin{bmatrix} \mathcal{C}_q(1) \\ \mathcal{C}_q(2) \end{bmatrix} = \left[\mathcal{C}_{q,1} \, \mathcal{C}_{q,2} \cdots \mathcal{C}_{q,N_s(q)}\right] \in \mathbb{C}^{2N_c \times N_s(q)L}$$
$$\mathcal{C}_{q,r} = \begin{bmatrix} c_{q,r}(1) & \cdots & \mathbf{0}_{L-1\times 1} \\ \vdots & \ddots & c_{q,r}(1) \\ c_{q,r}(N_c) & \ddots & \vdots \\ 0 & \ddots & c_{q,r}(N_c) \\ \vdots & \cdots & \mathbf{0}_{N_c-L+1\times 1} \end{bmatrix} \in \mathbb{C}^{2N_c \times L}, \quad (4)$$

with $c_{q,r}(n)$ defined from the original code sequences $c_q(n)$, $q = 1 \ldots Q$, setting to zero all the samples outside the rth symbol interval, i.e.

$$c_{q,r}(n) = \begin{cases} c_q(n) & (r-1)SF_q < n \leq rSF_q \\ 0 & \text{otherwise} \end{cases}. \quad (5)$$

Matrix $\mathcal{D}_q(2)$ is obtained as the $\left\lceil \frac{L-1}{SF_q} \right\rceil$ columns on the right of $\mathcal{C}_q(2)$, and contains the contribution from symbols transmitted prior to the observation interval.

Returning to the signal model in (1), $\mathbf{s} \in \mathbb{C}^{M_s \times 1}$ contains the set of received symbols, i.e.

$$\mathbf{s} = \left[\mathbf{s}_1^T \, \mathbf{s}_2^T \cdots \mathbf{s}_Q^T\right]^T \in \mathbb{C}^{M_s \times 1} \quad (6)$$
$$\mathbf{s}_q = \left[s_q\left(-\left\lceil \tfrac{L-1}{SF_q} \right\rceil + 1\right) \cdots s_q(MN_s(q))\right]^T \in \mathbb{C}^{M_s(q)\times 1},$$

where $M_s = \sum_{q=1}^{Q} M_s(q)$.

It is assumed that the components of the noise vector are circularly symmetric Gaussian distributed with zero mean and covariance $E\left[\mathbf{nn}^H\right] = \sigma^2 \mathbf{I}_{MN_c}$. This seems a reasonable approximation since we are concentrating on a single-user scenario. The results obtained here could in principle be generalized to the multi-channel estimation case, modifying the signal model in (1) to include the contribution from several users. This is, however, out of the scope of this paper.

3. ML CHANNEL ESTIMATION METHODS

In the following, we present some channel estimation methods based on the principle of Maximum Likelihood. We will see that different ways of modelling the presence of the unknown symbols, transmitted simultaneously with the training sequence, will lead to distinct types of estimators with different asymptotic performance. In particular, we will concentrate on three different types of estimators: a classical training-only (TO) estimator, which results from ignoring the presence of the unknown symbols; a CML estimator, based on a model in which unknown data are regarded as deterministic parameters; and a GML estimator, which arises from modelling them as Gaussian-distributed random variables.

3.1. Training-only Approach

The training only estimator disregards the presence of the traffic channels, which is equivalent to setting $\mathbf{s} = \mathbf{0}$ in (1). The ML estimator for this signal model can be expressed as

$$\hat{\mathbf{h}}_{to} = \left(\mathcal{T}^H \mathcal{T}\right)^{-1} \mathcal{T}^H \mathbf{x}. \quad (7)$$

In order to investigate the asymptotic performance of the channel estimator, we make the following assumptions:

(**As1**) The unknown symbols are circularly symmetric i.i.d. random variables with zero mean and unit variance: $E\left[s_q(i)s_p^*(j)\right] = \delta_{i-j}\delta_{p-q}$.

(**As2**) The spreading codes are circularly symmetric i.i.d. random variables with zero mean, variance $E\left[c_p(n)c_q^*(m)\right] = \alpha_q \delta_{p-q}\delta_{n-m}$ and independent of the unknown symbols.

(**As3**) The training sequences are circularly symmetric i.i.d. random variables with zero mean, variance $E\left[t(n)t^*(m)\right] = \alpha^t \delta_{n-m}$, and independent of the unknown symbols and the spreading sequences.

Proposition 1 *Under (*__As1__*,*__As2__*,*__As3__*), the training-only channel estimator $\sqrt{M}\hat{\mathbf{h}}_{to}$ is consistent and asymptotically (in M) circularly symmetric Gaussian-distributed with covariance matrix given by*

$$\mathbf{C}_{to} = \underbrace{\frac{\sigma^2}{\alpha^t N_c}\mathbf{I}_L}_{M\mathbf{B}_{to}} + \frac{\alpha}{\alpha^t N_c}\mathbf{R}_h \quad (8)$$
$$\{\mathbf{R}_h\}_{i,j} = \sum_{l=1}^{L-i+j} h^*(l)h(l+i-j), \quad i \geq j$$

and $\{\mathbf{R}_h\}_{i,j} = \{\mathbf{R}_h\}_{j,i}^*$, where α stands for the global power of the spreading codes allocating traffic data, $\alpha = \sum_{q=1}^{Q}\alpha_q$. The diagonal matrix $\mathbf{B}_{to}$ denotes the Training-only Cramér-Rao Bound for the asymptotic conditions specified.

Proof. See [1].

Defining now the effective Signal to Noise Ratio as

$$\gamma_x = \frac{\|\mathbf{h}\|^2\left(\alpha^t + \alpha\right)N_c}{\sigma^2}, \quad (9)$$

we can express the normalized covariance matrix as

$$\frac{\mathbf{C}_{to}}{\|\mathbf{h}\|^2} = \frac{1}{\gamma_x}\left(1+\frac{\alpha}{\alpha^t}\right)\mathbf{I}_L + \frac{1}{N_c}\left(\frac{\alpha}{\alpha^t}\right)\frac{\mathbf{R}_h}{\|\mathbf{h}\|^2}. \tag{10}$$

When the effective signal to noise ratio increases without bound the relative asymptotic covariance matrix tends to the constant value $\frac{1}{N_c}\left(\frac{\alpha}{\alpha^t}\right)\frac{\mathbf{R}_h}{\|\mathbf{h}\|^2}$. Due to the presence of the traffic channels ($\alpha \neq 0$) and the finite period of the code sequence ($N_c < \infty$) the estimator can never attain the Cramér-Rao bound, however high the signal to noise ratio might be.

The poor performance of this estimator can be overcome with an explicit modelling of the traffic channels in the signal model of (1). In particular, one can model the unknown data either as deterministic parameters (Conditional Approach) or as random variables (Unconditional Approach); see [2] and references therein. These two approaches will lead to two distinct estimators which, as shown in [3, 4, 5], do not perform equivalently.

3.2. Conditional (Deterministic) ML Approach

If we model the unknown data as deterministic parameters, the ML channel estimator $\hat{\mathbf{h}}_c$ can be obtained minimizing the following negative log-likelihood function

$$\eta_c\left(\mathbf{h},\sigma^2\right) = MN_c \log\left(\pi\sigma^2\right) + \frac{1}{\sigma^2}\left(\mathbf{x}-\mathcal{T}\mathbf{h}\right)^H \mathbf{P}_{\mathbf{G}}^{\perp}\left(\mathbf{x}-\mathcal{T}\mathbf{h}\right), \tag{11}$$

with $\mathbf{P}_{\mathbf{G}}^{\perp}$ denoting the orthogonal projection matrix onto the null space of the columns of $\mathbf{G}$. A closed expression of the CML channel estimator can be found in, e.g. [3].

Proposition 2 *Assume that (***As1***,* **As2***,* **As3***) hold. The normalized conditional channel estimator $\sqrt{M}\hat{\mathbf{h}}_c$ is consistent and asymptotically ($M \to \infty$) circularly symmetric Gaussian-distributed. If, in addition, the noise power, the period of the spreading codes and the corresponding Spreading Factors increase without bound at the same rate (σ^2, N_c, $SF_q \to \infty$) while their quotient remains constant, the asymptotic covariance is given by*[1]

$$\dot{\mathbf{C}}_c = M\dot{\mathbf{B}}_c + \frac{\|\mathbf{h}\|^2}{\gamma_x}\left(\frac{N_s}{\gamma_x}\right)\mathbf{P}_{\mathbf{h}}^{\perp} \tag{12}$$
$$M\dot{\mathbf{B}}_c = \frac{\|\mathbf{h}\|^2}{\gamma_x}\left[\mathbf{P}_{\mathbf{h}}^{\perp} + \left(1+\frac{\alpha}{\alpha^t}\right)\mathbf{P}_{\mathbf{h}}\right].$$

with $\dot{\mathbf{B}}_c$ the asymptotic conditional Cramér-Rao bound under the conditions specified, $\mathbf{P}_{\mathbf{h}} = \frac{\mathbf{h}\mathbf{h}^H}{\|\mathbf{h}\|^2}$, $\mathbf{P}_{\mathbf{h}}^{\perp} = \mathbf{I}_L - \mathbf{P}_{\mathbf{h}}$ and γ_x defined in (9). Furthermore, this result holds regardless of the statistical assumption about the unknown data.
Proof. See [1].

Two observations are in order. First, note that the covariance of the CML channel estimator does not depend on how the symbols are distributed across the different codes or what is the power associated with each code. Instead, it depends on the total number of transmitted symbols per code period N_s and the global power associated with the codes α. Second, we see from (12) that the conditional channel estimator is inefficient for finite values of the signal to noise ratio.

3.3. Gaussian ML Approach

According to the Gaussian ML Approach, symbols are modelled as complex mutually independent circularly symmetric Gaussian random variables with zero mean and unit covariance. A GML approach is preferred to a strictly Unconditional ML estimator (according to which symbols should be modelled with a discrete uniform distribution) because we seek to obtain methods based on second order statistics of the received signal only.

The negative log-likelihood function to be minimized can be expressed as

$$\eta_g\left(\sigma^2,\mathbf{h}\right) = \log\det\left(\pi\mathbf{C}_x\right) + \left(\mathbf{x}-\mathcal{T}\mathbf{h}\right)^H \mathbf{C}_x^{-1}\left(\mathbf{x}-\mathcal{T}\mathbf{h}\right), \tag{13}$$

where $\mathbf{C}_x = \mathbf{G}\mathbf{G}^H + \sigma^2\mathbf{I}_{MN_c}$ represents the temporal covariance matrix associated with the random component of the signal. A closed expression for the GML channel estimator can be found in [1].

Proposition 3 *Assume that (***As1, As2, As3***) hold. Assume further that the product $\alpha_q SF_q$ remains constant for all the transmitted codes.*[2] *Then, the GML channel estimator is consistent, efficient and asymptotically ($M \to \infty$) Gaussian-distributed. If, in addition, the noise power and the period of the spreading codes and the corresponding Spreading Factors increase without bound at the same rate (σ^2, N_c, $SF_q \to \infty$) while their quotient remains constant, the asymptotic covariance of the GML channel estimator is given by*

$$\dot{\mathbf{C}}_g = M\dot{\mathbf{B}}_g \tag{14}$$
$$M\dot{\mathbf{B}}_g = \frac{\|\mathbf{h}\|^2}{\gamma_x}\left[\frac{\gamma_s(q)+1}{\gamma_s(q)+\frac{\alpha^t}{\alpha^t+\alpha}}\mathbf{P}_{\mathbf{h}}^{\perp} + \left(1+\frac{\alpha}{\alpha^t}\right)\mathbf{P}_{\mathbf{h}}\right],$$

with $\dot{\mathbf{B}}_g$ the Cramér-Rao Bound under the asymptotic conditions specified and $\gamma_s(q)$ denoting Symbol Energy to Noise Power Spectral Density associated with channel q

$$\gamma_s(q) = \frac{\alpha_q\|\mathbf{h}\|^2 SF_q}{\sigma^2}. \tag{15}$$

Moreover, the same result is obtained regardless of the statistical assumption about the unknown symbols.
Proof. See [1].

Next, we compare the asymptotic covariances of the three methods under consideration. Their influence on the performance of the symbol detectors is analyzed in [6].

4. ANALYTIC COMPARISONS

From the expressions obtained in the last Section, it is easy to establish the following inequalities, which are asymptotically valid when M as well as SF_q and σ^2 increase without bound (the last two parameters at the same rate):

[1] Throughout the paper, a dot $[\dot{}]$ will be used upon covariances calculated under this double asymptotic limit.

[2] This assumption guarantees a constant reception quality for each of the transmitted codes.

$$M\dot{\mathbf{B}}_c \leq \dot{\mathbf{C}}_g = M\dot{\mathbf{B}}_g \leq M\dot{\mathbf{B}}_{to} \quad (16)$$

$$M\dot{\mathbf{B}}_g \leq \dot{\mathbf{C}}_c \quad (17)$$

$$\left.\begin{array}{ll} M\dot{\mathbf{B}}_{to} \geq \dot{\mathbf{C}}_c & \gamma_s(q) \geq \gamma_{th} \\ M\dot{\mathbf{B}}_{to} < \dot{\mathbf{C}}_c & \gamma_s(q) < \gamma_{th} \end{array}\right\}, \quad (18)$$

with $\gamma_{th} = \left(1 + \frac{\alpha}{\alpha^t}\right)^{-1}$ and where $\mathbf{A} \leq \mathbf{B}$ indicates that $\mathbf{B} - \mathbf{A}$ is positive semidefinite.

According to (16), under the asymptotic conditions specified, the semi-blind GML estimator performs better than the training-only method, while the inequality in (17) indicates that the conditional method performs worse than the Gaussian method. As a conclusion, the Gaussian Cramér-Rao bound can be interpreted, under the present assumptions and asymptotic limits, as the performance bound for both conditional and Gaussian methods, whereas the asymptotic Conditional Cramér-Rao bound can never be attained with second order approaches. All these conclusions are in perfect agreement with the results presented in [2] in the context of direction of arrival estimation.

It is finally observed from (18) that, surprisingly, the conditional semi-blind method can perform worse than its training-only counterpart at low values of the effective signal to noise ratio. The threshold symbol energy to noise power density γ_{th} establishes a limit over which a performance gain can be expected from using semi-blind conditional estimation schemes under our asymptotic assumptions. In any case, the values of γ_{th} will in practice be low enough to guarantee that the semi-blind conditional scheme performs better than the training-only estimator (values of the traffic to training power ratio are expected to be much higher than one in actual WCDMA systems).

In Figures 1 and 2, the trace of the asymptotic covariances normalized by the squared norm of $\mathbf{h}$ is represented as a function of the traffic to training power ratio (α/α_t) and the effective signal to noise ratio (γ_x) respectively. The relationships in (16)-(18) can be readily verified. It is finally worth stressing that, as shown in [1], the asymptotic expressions derived in this paper $\mathbf{C}_{to}, \dot{\mathbf{C}}_c, \dot{\mathbf{C}}_g$ are very close to the actual ones for spreading factors $SF_q \geq 16$.

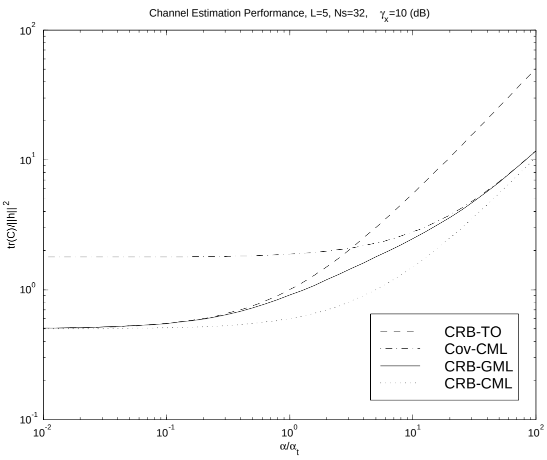

Figure 1: Asymptotic covariances as a function of α/α_t.

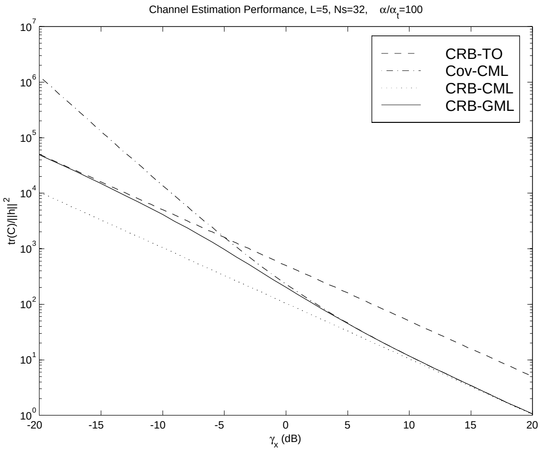

Figure 2: Asymptotic covariances as a function of γ_x.

5. CONCLUSIONS

We have derived asymptotic expressions describing the performance of three different channel estimation algorithms in a pilot sequence-aided multi-rate CDMA framework. For the classic training-only channel estimation method, we have been able to describe the mean asymptotic behavior as the number of spreading periods (M) increases without bound. After investigation of the resulting covariance matrix we have shown that the performance of the algorithm tends to saturate as the effective signal to noise ratio (γ_x) increases. Semi-blind techniques have been proposed as means to overcome this effect. The performance of two different semi-blind approaches has been evaluated under asymptotic conditions in the number of spreading periods M when both the spreading factor and the noise power tend to infinity. These results have finally been used to establish the relationship among different techniques under the asymptotic conditions considered.

6. REFERENCES

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