DIRECTION OF ARRIVAL ESTIMATION IN PARTLY CALIBRATED TIME-VARYING SENSOR ARRAYS

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ABSTRACT

We consider the direction finding problem in time-varying arrays composed of identically oriented subarrays displaced by unknown vector translations. A new eigenstructure-based estimator is proposed for such a class of partly calibrated sensor arrays.

1. INTRODUCTION

The problem of direction finding using time-varying sensor arrays is important in several practical applications [1]. Existing solutions to this problem [1]-[2] require the exact knowledge of sensor positions during the whole observation time. However, there exist many situations where such a knowledge remains unavailable because of prohibitively high complexities of calibration techniques and fast variations of array geometry.

In this paper, we present a new eigenstructure-based approach to direction finding with partly calibrated arrays which may involve several calibrated subarrays displaced by unknown time-varying vector translations. Our method (referred to as the RAnk REduction (RARE) estimator) enjoys simple implementation which entails computing the eigendecomposition of the sample array covariance matrix and polynomial rooting.

2. PROBLEM FORMULATION

Consider an array of M omnidirectional sensors which receives $L < M$ narrowband signals impinging from the sources with the unknown Directions Of Arrival (DOA's) $\theta_1, \dots, \theta_L$. The parameter L is assumed to be known [3]. Let this array consist of K identically oriented linear subarrays whose interelement spacings are integer multiples of the known *shortest baseline* Δ. An example of such an array composed of three subarrays is shown in Fig. 1. The geometry of each subarray is assumed to be known, whereas the *inter-subarray displacements* are assumed to be unknown. Note that in Sections 3 and 4.1, these displacements are considered to be time-invariant, whereas in Section 4.2 the case of unknown time-varying inter-subarray displacements will be treated. Let $M_k \geq 1$ be the number of sensors of the kth subarray, so that $M = \sum_{k=1}^{K} M_k$. We stress that M_k may take different values for various subarrays.

For the sake of simplicity, it is convenient to define each subarray by means of a certain planar translation of a part of sensors of an M-element *nominal* (virtual) uniform linear array (ULA). This representation is illustrated in Fig. 2, where the second and third subarrays of Fig. 1 are interpreted as a result of two unknown vector translations $\boldsymbol{\xi}_k$, $k = 1, 2$. In the general case of K subarrays, the $K-1$ translation vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_{K-1}$ are required to determine the array geometry ($\boldsymbol{\xi}_0 = \mathbf{0}$).

The problem is to estimate the DOA vector

$$\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_L]^T$$

where $(\cdot)^T$ denotes the transpose.

3. SIGNAL MODEL

Using the nominal ULA representation described above, it can be readily shown that the narrowband model for the $M \times 1$ steering vector may be written as

$$\boldsymbol{a}(\theta, \boldsymbol{\alpha}) = \boldsymbol{Q}(\theta)\boldsymbol{T}\boldsymbol{h}(\theta, \boldsymbol{\alpha}) \tag{1}$$

where the $2(K-1) \times 1$ vector $\boldsymbol{\alpha} = \mathrm{vec}\{\boldsymbol{\Omega}\}$, the $(K-1) \times 2$ matrix $\boldsymbol{\Omega} = \left[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_{K-1}\right]^T$, and $\mathrm{vec}\{\cdot\}$ is the operator stacking the columns of a matrix on top of each other. The vector $\boldsymbol{\alpha}$ combines all unknown inter-subarray displacement parameters,

$$\boldsymbol{h}(\theta, \boldsymbol{\alpha}) = \left[1, \exp\{j(2\pi/\lambda)\boldsymbol{\xi}_1^T \boldsymbol{\phi}\} \dots, \exp\{j(2\pi/\lambda)\boldsymbol{\xi}_{K-1}^T \boldsymbol{\phi}\}\right]^T$$

$$\begin{aligned} \boldsymbol{Q}(\theta) \quad = \quad & \mathrm{diag}\,\{1, \exp\{j(2\pi/\lambda)\Delta \sin\theta\} \\ & \dots, \exp\{j(M-1)(2\pi/\lambda)\Delta\sin\theta\}\}^T \end{aligned} \tag{2}$$

$\boldsymbol{\phi} = [\sin\theta, \cos\theta]^T$, $\boldsymbol{\xi}_k = [\xi_{x,k}, \xi_{y,k}]^T$, and λ is the wavelength. The $M \times K$ *selection matrix* $\boldsymbol{T}$ consists of zeros and ones and "distributes" the sensors of the nominal ULA among the subarrays. That is, the (m, k)th element of $\boldsymbol{T}$ is equal to one if, after the translation by $\boldsymbol{\xi}_{k-1}$, the mth virtual ULA sensor becomes a part of the kth subarray, and equal to zero otherwise.

For example, for the array configuration depicted in Fig. 2,

$$\boldsymbol{T}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{3}$$

Using (1), the array snapshots can be modeled as

$$\boldsymbol{x}(t) = \boldsymbol{A}(\boldsymbol{\theta}, \boldsymbol{\alpha})\boldsymbol{s}(t) + \boldsymbol{n}(t) \tag{4}$$

where $\boldsymbol{A}(\boldsymbol{\theta}, \boldsymbol{\alpha}) = [\boldsymbol{a}(\theta_1, \boldsymbol{\alpha}), \dots, \boldsymbol{a}(\theta_L, \boldsymbol{\alpha})]$ is the $M \times L$ direction matrix, $\boldsymbol{s}(t)$ is the $L \times 1$ vector of the signal waveforms, and $\boldsymbol{n}(t)$ is the $M \times 1$ vector of white sensor noise.

The sample covariance matrix is given by

$$\hat{\boldsymbol{R}} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{x}(t)\boldsymbol{x}^H(t) \tag{5}$$

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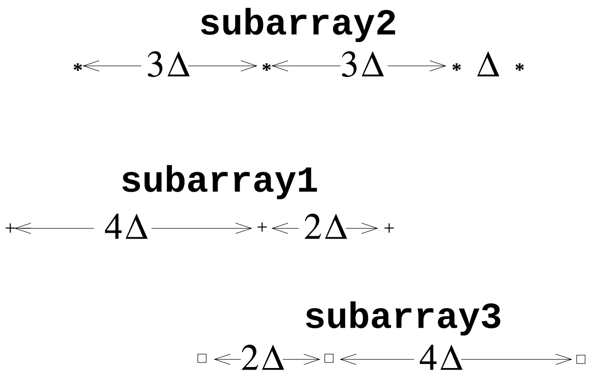

Fig. 1. A particular example of the considered type of sensor array: first subarray ($+$), second subarray ($*$), third subarray ($\Box$).

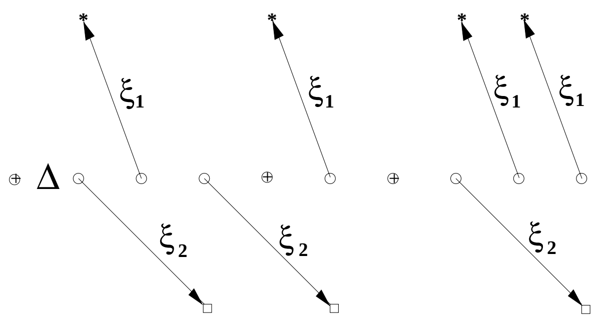

Fig. 2. An interpretation of the array structure of Fig. 1 using the concept of a nominal ULA (o) and two vector displacements $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$.

where $(\cdot)^H$ denotes the Hermitian transpose. The eigendecomposition of (5) yields

$$\hat{\boldsymbol{R}} = \hat{\boldsymbol{E}}_S \hat{\boldsymbol{\Lambda}}_S \hat{\boldsymbol{E}}_S^H + \hat{\boldsymbol{E}}_N \hat{\boldsymbol{\Lambda}}_N \hat{\boldsymbol{E}}_N^H \tag{6}$$

where the $L \times L$ and $(M-L)\times(M-L)$ diagonal matrices $\hat{\boldsymbol{\Lambda}}_S$ and $\hat{\boldsymbol{\Lambda}}_N$ contain the L and $M-L$ signal and noise-subspace eigenvalues, respectively, and the columns of the $M \times L$ and $M \times (M-L)$ matrices $\hat{\boldsymbol{E}}_S$ and $\hat{\boldsymbol{E}}_N$ contain the signal and noise-subspace eigenvectors, respectively. Note that (6) is the sample estimate of the true covariance matrix

$$\boldsymbol{R} = \mathrm{E}\{\boldsymbol{x}(t)\boldsymbol{x}^H(t)\} = \boldsymbol{E}_S \boldsymbol{\Lambda}_S \boldsymbol{E}_S^H + \boldsymbol{E}_N \boldsymbol{\Lambda}_N \boldsymbol{E}_N^H$$

4. DOA ESTIMATION

4.1. Time-Invariant Arrays

We make the following assumptions, which are required for the formulation of our technique:
(A1) The selection matrix $\boldsymbol{T}$ has a full column rank and at most one nonzero entry in each row.
(A2) The number of subarrays K is chosen so that $K \leq M - L$.

We start our derivation from the consideration of the conventional spectral MUSIC algorithm which estimates the signal DOA's from the L deepest minima of the function [3]

$$f_{\mathrm{MUSIC}}(\theta, \boldsymbol{\alpha}) = \boldsymbol{a}^H(\theta, \boldsymbol{\alpha}) \hat{\boldsymbol{E}}_N \hat{\boldsymbol{E}}_N^H \boldsymbol{a}(\theta, \boldsymbol{\alpha}) \tag{7}$$

In particular, in the ideal case of exactly known $\boldsymbol{R}$, the DOA's can be found from the equation [3]

$$\boldsymbol{a}^H(\theta, \boldsymbol{\alpha}) \boldsymbol{E}_N \boldsymbol{E}_N^H \boldsymbol{a}(\theta, \boldsymbol{\alpha}) = 0 \tag{8}$$

However, the vector parameter $\boldsymbol{\alpha}$ is unknown and, therefore, the minimization of (8) requires an exhaustive $2(K-1)+1$-dimensional search which becomes totally impractical for $K > 1$. Using (1), we can rewrite (7) as

$$\begin{aligned} f_{\mathrm{MUSIC}}(\theta, \boldsymbol{\alpha}) &= \boldsymbol{h}^H(\theta, \boldsymbol{\alpha}) \boldsymbol{T}^T \boldsymbol{Q}^H(z) \hat{\boldsymbol{E}}_N \hat{\boldsymbol{E}}_N^H \boldsymbol{Q}(z) \boldsymbol{T} \boldsymbol{h}(\theta, \boldsymbol{\alpha}) \\ &= \boldsymbol{h}^H(\theta, \boldsymbol{\alpha}) \hat{\boldsymbol{B}}(z) \boldsymbol{h}(\theta, \boldsymbol{\alpha}) \end{aligned} \tag{9}$$

where

$$\hat{\boldsymbol{B}}(z) = \boldsymbol{T}^T \boldsymbol{Q}^H(z) \hat{\boldsymbol{E}}_N \hat{\boldsymbol{E}}_N^H \boldsymbol{Q}(z) \boldsymbol{T} \tag{10}$$

is the $K \times K$ Hermitian matrix, and $z = \exp\{j(2\pi/\lambda)\Delta \sin\theta\}$. An important observation following from (9) and (10) is that the vector parameter $\boldsymbol{\alpha}$ is contained in $\boldsymbol{h}(\theta, \boldsymbol{\alpha})$ only, so that the matrix $\hat{\boldsymbol{B}}(z)$ is independent of $\boldsymbol{\alpha}$. Also, it is worth noting that the matrix $\boldsymbol{Q}$ in (9) and (10) is reformulated in terms of z, so that

$$\boldsymbol{Q}(z) = \mathrm{diag}\left\{1, z, \ldots, z^{M-1}\right\} \tag{11}$$

In the ideal case of exactly known $\boldsymbol{R}$, we can rewrite equation (8) as

$$\boldsymbol{h}^H(\theta, \boldsymbol{\alpha}) \boldsymbol{B}(z) \boldsymbol{h}(\theta, \boldsymbol{\alpha}) = 0 \tag{12}$$

where

$$\boldsymbol{B}(z) = \boldsymbol{T}^T \boldsymbol{Q}^H(z) \boldsymbol{E}_N \boldsymbol{E}_N^H \boldsymbol{Q}(z) \boldsymbol{T}$$

Note that $\boldsymbol{h}(\theta, \boldsymbol{\alpha}) \neq \boldsymbol{0}$ and, therefore, (12) holds true iff

$$\mathrm{rank}\{\boldsymbol{B}(z)\} < K \tag{13}$$

or, equivalently, iff the polynomial

$$P(z) = \det\{\boldsymbol{B}(z)\} = 0 \tag{14}$$

Note that according to assumptions A1 and A2 the matrix $\boldsymbol{B}(z)$ will in the general case be of full rank K. However, its rank would reduce if z becomes equal to one of the roots of $P(z)$. Hence, *the signal DOA's can be obtained by rooting the polynomial $P(z)$ without needing any knowledge of the inter-subarray displacement parameters $\boldsymbol{\alpha}$!*

Now, we apply these results to the realistic case when only the sample covariance matrix $\hat{\boldsymbol{R}}$ is known. In this case, we can formulate the following algorithm, which is referred to as the RAnk REduction (RARE) estimator:

- *Step 1.* Compute the eigendecomposition of $\hat{\boldsymbol{R}}$ and find $\hat{\boldsymbol{E}}_N$.
- *Step 2.* Root the polynomial $\hat{P}(z) = \det\{\hat{\boldsymbol{B}}(z)\}$. Find the signal DOA estimates $\hat{\theta}_l$, $l = 1, 2, \ldots, L$ from the L *signal roots*[1] $\hat{z}_l$, $l = 1, 2, \ldots, L$ located inside the unit circle.

Remark 1: The polynomial root-finding step is similar to that of root-MUSIC [4]. However, the forms of the RARE and root-MUSIC polynomials are completely different. Furthermore, the application of root-MUSIC is restricted by the fully calibrated ULA

[1] The L roots closest to the unit circle are referred to as the signal roots.

case, whereas RARE is applicable to the case of nonuniform partly calibrated arrays.

Remark 2: Interestingly, the idea behind the RARE algorithm is related to the approach [5] which extends root-MUSIC to diversely oriented velocity hydrophone ULA's. Also, the criterion similar to (14) was used in [6] to extend root-MUSIC to the case of fully-calibrated arrays with multiple invariances. However, it is important to stress that our problem and signal model are completely different from that exploited in [5] and [6].

Remark 3: Fast algorithms for computing the coefficients of $\hat{P}(z)$ are available, so that the major computational load of RARE is due to the eigendecomposition of $\hat{\boldsymbol{R}}$ (see [7] for details).

Remark 4: In the particular case $K = 1$, the array becomes a fully calibrated ULA, and we have that $\boldsymbol{T} = [1, 1, \ldots, 1]^T$ and, therefore, $\hat{\boldsymbol{B}}(z)$ becomes a scalar. In this case, the RARE polynomial is identical to the conventional root-MUSIC polynomial [4], i.e.

$$\hat{P}(z)\Big|_{K=1} = \boldsymbol{a}^H(z)\hat{\boldsymbol{E}}_N\hat{\boldsymbol{E}}_N^H\boldsymbol{a}(z) = f_{\text{MUSIC}}(z) \qquad (15)$$

where $\boldsymbol{a}(z) = [1, z, \ldots, z^{M-1}]^T$.

4.2. Time-Varying Arrays

In this section, the case of unknown time-varying inter-subarray displacements $\boldsymbol{\alpha} = \boldsymbol{\alpha}(t)$ is considered. Similarly to [1] and [2], we assume that *the signal DOA's remain fixed within the whole observation interval of N snapshots*. Let us divide this interval into J nonoverlapping subintervals of the length $\tilde{N} = N/J$ and assume w.l.g. *the piecewise time-invariance of the inter-subarray displacements within each of such subintervals.* In other words, the subinterval length $\tilde{N}$ is assumed to be so short that the variation of the array geometry remains negligible within a subinterval. Applying RARE to each subinterval, we obtain the polynomials

$$\hat{P}_i(z) = \det\{\hat{\boldsymbol{B}}_i\} \qquad i = 1, 2, \ldots, J \qquad (16)$$

where

$$\hat{\boldsymbol{B}}_i(z) = \boldsymbol{T}_i^T\boldsymbol{Q}^H(z)\hat{\boldsymbol{E}}_{N,i}\hat{\boldsymbol{E}}_{N,i}^H\boldsymbol{Q}(z)\boldsymbol{T}_i$$

Here, $\boldsymbol{T}_i$ and $\hat{\boldsymbol{E}}_{N,i}$ are the selection and noise-subspace eigenvector matrices, respectively, computed at the ith observation subinterval. It is worth noting that we not only allow the inter-subarray displacements to vary in a completely unknown way between any two different time subintervals, but, due to the time-varying structure of the matrix $\boldsymbol{T}_i$, it also becomes possible to rearrange sensors dynamically between subarrays (for example, partition or merge subarrays), provided that the assumptions A1 and A2 are not violated.

To combine the results of the application of RARE to each observation subinterval, let us average the polynomials $\hat{P}_i(z)$ over the whole observation length

$$\mathcal{P}(z) = \sum_{i=1}^{J}\hat{P}_i(z) \qquad (17)$$

Then, the signal DOA's can be obtained from the signal roots of $\mathcal{P}(z)$. Clearly, the averaging operation will enhance the signal roots and improve the performance compared to that at each particular subinterval (see [7] for the formal proof of this fact). In Section 6, it will be demonstrated by computer simulations that the averaged RARE algorithm achieves the performance nearly identical to the corresponding Cramér-Rao bound (CRB).

5. CRAMÉR-RAO BOUNDS

Let in the case of time-varying arrays the observations satisfy the following deterministic model

$$\boldsymbol{x}_i(t) \sim \mathcal{N}\left\{\boldsymbol{A}(\boldsymbol{\theta}, \boldsymbol{\alpha}_i)\boldsymbol{s}_i(t), \sigma^2\boldsymbol{I}\right\} \qquad (18)$$

where $\boldsymbol{x}_i(t) = [x_{i,1}(t), \ldots, s_{i,M}(t)]^T$ and $\boldsymbol{s}_i(t) = [s_{i,1}(t), \ldots, s_{i,L}(t)]^T$ are the observation and source waveform vectors, respectively, corresponding to the tth sample of the ith subinterval. As before, the inter-subarray displacements are treated as *unknown parameters* together with the signal DOA's, deterministic source waveforms, and the sensor noise variance. Then, the following closed-form expression for the DOA-related block of the CRB matrix can be obtained

$$\begin{aligned}
\mathbf{CRB}_{\boldsymbol{\theta\theta}} &= \frac{\sigma^2}{2}\left\{\sum_{i=1}^{J}\left(\boldsymbol{F}_i - \boldsymbol{M}_i\boldsymbol{G}_i^{-1}\boldsymbol{M}_i^T\right)\right\}^{-1} \qquad (19)\\
\boldsymbol{A}_i &= \boldsymbol{A}(\boldsymbol{\theta}, \boldsymbol{\alpha}_i) \triangleq [\boldsymbol{a}_{i,1}, \ldots, \boldsymbol{a}_{i,L}]\\
\boldsymbol{F}_i &= \sum_{t=1}^{\tilde{N}}\text{Re}\left\{\boldsymbol{D}_i^H(t)\boldsymbol{\Pi}_{\boldsymbol{A}_i}^{\perp}\boldsymbol{D}_i(t)\right\}\\
\boldsymbol{M}_i &= \sum_{t=1}^{\tilde{N}}\text{Re}\left\{\boldsymbol{D}_i^H(t)\boldsymbol{\Pi}_{\boldsymbol{A}_i}^{\perp}\boldsymbol{H}_i(t)\right\}\\
\boldsymbol{G}_i &= \sum_{t=1}^{\tilde{N}}\text{Re}\left\{\boldsymbol{H}_i^H(t)\boldsymbol{\Pi}_{\boldsymbol{A}_i}^{\perp}\boldsymbol{H}_i(t)\right\}\\
\boldsymbol{\Pi}_{\boldsymbol{A}_i}^{\perp} &= \boldsymbol{I} - \boldsymbol{A}_i(\boldsymbol{A}_i^H\boldsymbol{A}_i)^{-1}\boldsymbol{A}_i^H\\
\boldsymbol{D}_i(t) &= \left[\frac{\delta\boldsymbol{a}_{i,1}}{\delta\theta_1}s_{i,1}(t), \ldots, \frac{\delta\boldsymbol{a}_{i,L}}{\delta\theta_L}s_{i,L}(t)\right]\\
\boldsymbol{H}_i(t) &= \left[\tilde{\boldsymbol{H}}_i(t), \bar{\boldsymbol{H}}_i(t)\right]\\
\tilde{\boldsymbol{H}}_i(t) &= j(2\pi/\lambda)\tilde{\boldsymbol{T}}_i \odot (\boldsymbol{A}_i\boldsymbol{\Phi}\boldsymbol{s}_i(t)\mathbf{1}^T)\\
\bar{\boldsymbol{H}}_i(t) &= j(2\pi/\lambda)\tilde{\boldsymbol{T}}_i \odot (\boldsymbol{A}_i\boldsymbol{\Psi}\boldsymbol{s}_i(t)\mathbf{1}^T)
\end{aligned}$$

where the matrix $\tilde{\boldsymbol{T}}_i$ is formed from $\boldsymbol{T}_i$ by deleting its first column, $\mathbf{1}$ is the $(K-1)\times 1$ vector of ones, $\boldsymbol{\Phi} = \text{diag}\{\sin\theta_1, \ldots, \sin\theta_L\}$, $\boldsymbol{\Psi} = \text{diag}\{\cos\theta_1, \ldots, \cos\theta_L\}$, and $\odot$ is the Schur-Hadamard matrix product.

In the case of time-invariant arrays, the CRB can be obtained from (19) by assuming $J = 1$ and $\tilde{N} = N$. This bound is given by

$$\mathbf{CRB}_{\boldsymbol{\theta\theta}} = \frac{\sigma^2}{2}\left\{\boldsymbol{F} - \boldsymbol{M}\boldsymbol{G}^{-1}\boldsymbol{M}^T\right\}^{-1} \qquad (20)$$

where the matrices $\boldsymbol{A}$, $\boldsymbol{T}$, $\boldsymbol{F}$, $\boldsymbol{M}$, and $\boldsymbol{G}$ become independent of the index i. Proofs of equations (19) and (20) are presented in [7].

Comparing (20) to the CRB on DOA estimation in the case of a fully calibrated array [3], we obtain that the latter bound is given by $\mathbf{CRB}_{\text{C},\boldsymbol{\theta\theta}} = \frac{\sigma^2}{2}\boldsymbol{F}^{-1}$. It can be readily proven that $\boldsymbol{M}\boldsymbol{G}^{-1}\boldsymbol{M}^T$ is nonnegative definite. Therefore, from (20) we have $\mathbf{CRB}_{\boldsymbol{\theta\theta}} \geq \mathbf{CRB}_{\text{C},\boldsymbol{\theta\theta}}$. This result will be verified by a numerical example in the next section.

6. SIMULATIONS

We consider an array of $M = 10$ sensors which includes three subarrays of $M_1 = 6$, $M_2 = 3$, and $M_3 = 1$ sensors, respectively.

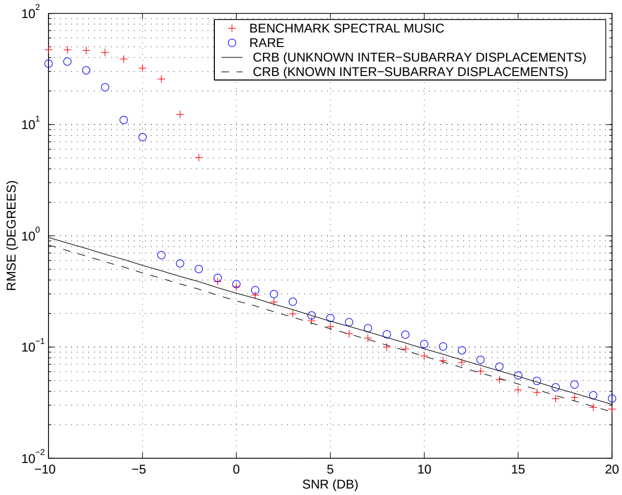

Fig. 3. RMSE's of benchmark spectral MUSIC, RARE, and the CRB's versus the SNR. $N = 100$.

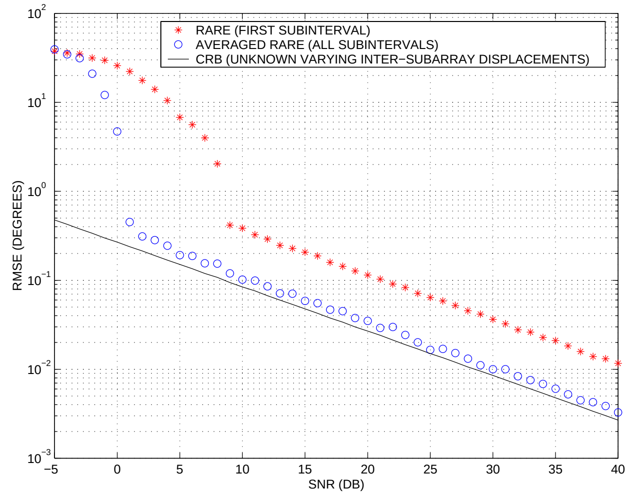

Fig. 4. RMSE's of RARE and averaged RARE, and the CRB versus the SNR. $N = 100$, ten subintervals, $\tilde{N} = 10$.

The first subarray corresponds to the sensors # 1, 3, 5, 7, 9, and 10 of the nominal ULA with the interelement spacing $\Delta = \lambda/2$. The second subarray corresponds the sensors # 2, 4, and 6 of the same ULA translated by $\boldsymbol{\xi}_1$, and the third subarray involves a single sensor corresponding to the sensor # 8 of the virtual ULA translated by $\boldsymbol{\xi}_2$. We assume two uncorrelated equi-powered sources with the DOA's $\theta_1 = 5°$ and $\theta_2 = 11°$. The number of snapshots is $N = 100$, the Signal to Noise Ratio (SNR) is varied, and all results are averaged over 100 simulation runs.

In the first example, the time-invariant array geometry is assumed, with the fixed displacements $\boldsymbol{\xi}_1 = [7.56\Delta, 25.43\Delta]$ and $\boldsymbol{\xi}_2 = [0.93\Delta, -12.27\Delta]$. In the second example, the time-varying array case is considered, where the interval of $N = 100$ snapshots is divided into ten nonoverlapping subintervals of the length $\tilde{N} = 10$. For each subinterval, the elements of the displacement vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ have been drawn from the uniform random generator with the mean zero and the standard deviation 30Δ. In Fig. 3, the DOA estimation Root-Mean-Square Errors (RMSE's) of RARE and the so-called *benchmark* spectral MUSIC algorithm are shown versus the SNR for the first example. We stress that, in contrast to RARE, benchmark MUSIC exploits the *complete knowledge of array geometry*, including the knowledge of inter-subarray displacements. Additionally, two different deterministic CRB's are displayed in Fig. 3: the first one is the conventional bound derived under the assumption of the full knowledge of the array geometry [3], whereas the second one corresponds to the case of unknown inter-subarray displacements and is given by (20).

Fig. 4 displays the RMSE's of the averaged RARE algorithm (taking the average (17) of ten polynomials corresponding to all subintervals) and the unaveraged RARE technique (using only the single polynomial $P_1(z)$ which corresponds to the first subinterval) versus SNR for the second example. Additionally, the deterministic CRB is shown. The latter bound is computed using (19) for the time-varying array with unknown inter-subarray displacements.

From Fig. 3 we see that in the time-invariant array case, RARE performs asymptotically close to the benchmark spectral MUSIC. However, RARE clearly outperforms MUSIC in having a remarkably lower SNR threshold. It is important to note that, in contrast to MUSIC, *RARE does not require any knowledge of inter-subarray displacements and has much simpler implementation*. Fig. 4 clearly demonstrates that in the time-varying array case, the proposed averaging of RARE polynomials appears to be a *coherent operation* since the performance of averaged RARE is very close to the corresponding CRB.

7. REFERENCES

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