OPTIMAL SIGNAL DESIGN AND DETECTION FOR FAST FADING CHANNELS

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ABSTRACT

In this paper, we characterize optimal detection and signaling (in terms of signal-to-noise ratio) for a fast fading channel when the autocorrelation function of the channel variation is known. We find that the well known matched filter and Rake receiver are limiting cases of our result. Also, we provide a procedure that finds a pair of signals for digital communication. With land mobile fading channel, the signals obtained with our procedure have a performance significantly better than traditional flat-top pulse in terms of probability of error rate.

1. INTRODUCTION

In this paper, we address the problem of designing a signaling waveform for optimum transmission through a fast fading channel, where the optimum is with respect to maximum signal to noise ratio, under the assumption that the autocorrelation function of the channel variation function is known. Fig. 1 depicts the system, a generic system for fading channels with a correlation detector. We also characterize the optimal receiver for such a signal.

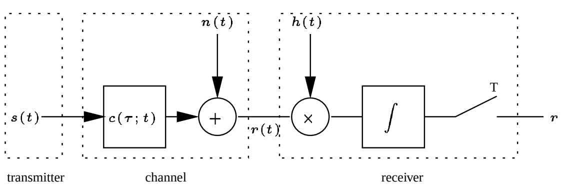

Fig. 1. System model.

Up to the present, research about fading channels has been performed largely under the slow fading assumption, i.e., the channel is considered to be constant for up to a few symbol times. Under this assumption, a channel locally behaves as an additive-white-Gaussian noise (AWGN) channel. It is generally accepted that matched filter detectors produce good performance. However, when the slow fading assumption is lifted, the optimal detector is not simply a matched filter. Fast fading causes variations within the symbol intervals so that the model applied to slow fading becomes inaccurate [1, 2]. The signal-to-noise ratio (SNR) in the receiver statistic r depends on the channel properties and the functions $s(t)$ and $h(t)$: Unlike un-faded transmission through a Gaussian channel, the transmitted waveform $s(t)$, rather than just its energy, has bearing on the SNR performance, and the optimal receiver signal $h(t)$ is not necessarily equivalent to $s(t)$.

The study can be largely simplified when we separate the fast fading and the frequency selectiveness aspects of the fading channel. When we assume frequency non-selectiveness, the channel $c(\tau;t)$ becomes simple multiplicative channel i.e., $r(t) = \alpha(t)s(t) + n(t)$, with $\alpha(t)$ being a zero-mean stationary complex Gaussian signal and $n(t)$ being complex AWGN with a power-spectral-density (PSD) $N_0/2$. The study maybe extended for frequency selective fading channels by introducing multiple paths to the channel model, i.e.,

$$r(t) = \sum_i \xi_i \alpha_i(t) s(t-\phi_i) + n(t), \qquad (1)$$

where $\alpha_i(t)$ is normalized so that $E[|\alpha_i|^2] = 1$, and ξ_i is the path strength and ϕ_i is the path delay for i^{th} path.

We will derive our results with single path situation herein. However, our discussion is readily extended for multipath fading channels. We first characterize optimal detectors for known transmitted signal, then we study the optimal signaling in the sense of maximizing the SNR of the received signal.

2. OPTIMAL DETECTION OF KNOWN SIGNALS

If we do not assume perfect synchronization, when a known signal $s_0(t)$ is transmitted, the received signal at the output of the detector filter is [1]

$$r = \int [\alpha(t)s_0(t-\mu) + n(t)]h(t)\,dt,$$

where μ is a random delay with density function of $p(\mu)$. The corresponding signal and noise components at the output of the detecting filter are $r_s = \int \alpha(t)s_0(t-\mu)h(t)\,dt$ and $r_n = \int n(t)h(t)\,dt$. Assuming $h(t)$ has unit energy, we have $E(|r_n|^2) = N_0/2$. The variance of the signal part is

$$\begin{aligned}
&E(|r_s|^2) \\
&= E\{\iint [\alpha(\tau)s_0(\tau-\mu)]h(\tau)[\alpha^*(\lambda)s_0^*(\lambda-\mu)]h^*(\lambda)\,d\tau\,d\lambda\} \\
&= \iint R_\alpha(\tau,\lambda)\Psi(\tau,\lambda)h(\tau)h^*(\lambda)\,d\tau\,d\lambda \overset{\text{def}}{=} F(h),
\end{aligned}$$

where $R_\alpha(\tau,\lambda) = E[\alpha(\tau)\alpha^*(\lambda)]$ and $\Psi(\tau,\lambda) = E[s(\tau-\mu)s^*(\lambda-\mu)] = \int s_0(\tau-\mu)s_0^*(\lambda-\mu)p(\mu)\,d\mu$.

To maximize $F(h)$ with constraint of $G(h) = \int h(t)h^*(t)\,dt = 1$, we form the Lagrangian $J(h) = F(h) - \gamma G(h)$, where γ is a Lagrange multiplier. We take a variation with respect to h (as defined in the appendix 1) of the Lagrangian and set it to zero, i.e.,

$$\frac{\delta}{\delta h}J(h_0) = \frac{\delta}{\delta h}F(h_0) - \gamma\frac{\delta}{\delta h}G(h_0) = 0 \qquad (2)$$

[1] Throughout this paper, integrals stated without limits are assumed to be over the interval $(-\infty,\infty)$.

where $h_0(t)$ is an extremum of $F(h)$. We have $\frac{\delta G(h_0)}{\delta h}(\tau) = h_0^*(\tau)$ and $\frac{\delta F(h_0)}{\delta h}(\tau) = \int R_\alpha(\tau,\lambda)\Psi(\tau,\lambda)h_0^*(\lambda)d\lambda$. From (2),

$$\gamma h_0^*(\tau) = \int R_\alpha(\tau,\lambda)\Psi(\tau,\lambda)h_0^*(\lambda)\,d\lambda. \tag{3}$$

For this h_0, $E(|r_s|^2) = \iint R_\alpha(\tau,\lambda)\Psi(\tau,\lambda)h_0(\tau)h_0^*(\lambda)d\tau d\lambda = \gamma$. We conclude that for maximum SNR, γ is the largest eigenvalue of $R_\alpha(\tau,\lambda)\Psi(\tau,\lambda)$, and $h_0^*(t)$ is the corresponding eigenfunction.

For perfect synchronization, i.e., $p(\mu) = \delta(\mu)$, it is interesting to point out that for the case of a single non-fading path, i.e., $R_\alpha(\tau,\lambda) = R$ (constant) for all τ and λ, $\gamma h_0^*(\tau) = \int R_\alpha(\tau,\lambda)$ $s_0(\tau)s_0^*(\lambda)h_0(\lambda)\,d\lambda = s_0(\tau)\int Rs_0^*(\lambda)h_0^*(\lambda)\,d\lambda = Cs_0(\tau)$, where $C \in \mathbb{C}$. This is the familiar matched filter. On the other hand, for the case of non-fading multiple paths, using (1), $\gamma h_0^*(\tau) = \sum_i\{|\xi_i|^2\int Rs_0^*(\lambda-\phi_i))h_0(\lambda)\,d\lambda\}s_0(\tau-\phi_i) = \sum_i C_i s_0(\tau-\phi_i)$, which is the rake receiver.

3. OPTIMAL SIGNALING (MAXIMIZING SNR)

Observing that $r = \int \alpha(t)s(t)\,h(t)\,dt$ is symmetric in $s(t)$ and $h(t)$, we can characterize the $s(t)$ that maximizes the SNR for a given $h(t)$ with the same argument above. This leads to an iterative algorithm: We find the optimal $h(t)$ for a given $s(t)$ then turn around and find the optimal $s(t)$ for this $h(t)$.

Algorithm 1: Initialize: Randomly select an initial $s_0(t)$.

1. Set $h_0(t)$ so that $\gamma_1 h_0^*(t) = \int R_\alpha(\tau,\lambda)\Psi(\tau,\lambda)h_0^*\,d\lambda$ where γ_1 is the largest eigenvalue, and $h_0^*(t)$ is the corresponding eigenfunction.
2. Set $s_0(t)$ so that $\gamma_0 s_0^*(t) = \int R_\alpha(\tau,\lambda)\Phi(\tau,\lambda)s_0^*(\lambda)\,d\lambda$, where $h_0(t)$ is that found in step 1, $\Phi(\tau,\lambda) = \int h_0(\tau+\mu)h_0^*(\lambda+\mu)p(\mu)\,d\mu$, and γ_0 is the largest eigenvalue and $s_0^*(t)$ is the corresponding eigenfunction.
3. Repeat steps 1 and 2 until convergence.

The integral equations can be approximated using discrete samples and matrix eigensolvers (e.g., in MATLAB). Since only the largest eigenvalue(s)/vector(s) is (are) needed, the power method can be used. It is not necessary to store the entire kernel as a matrix, since $\Psi(\tau,\lambda)$ constructed by a single vector, $R_\alpha(\tau,\lambda)$ is also symmetric Toeplitz, and hence is also characterized by a single vector. This allows higher numbers of samples (finer resolution) to be computed with modest storage requirements.

We have implemented this algorithm with discrete samples, $\Delta T = 1/128$sec, of $R_\alpha(\tau) = KJ_0(2\pi B_D\tau)$ from a land mobile model under Rayleigh fading, where K is a constant, and B_D is the maximum Doppler shift [3]. In the following examples, we use $B_D = 4.99744$Hz, that can be obtained when we are communicating with 900MHz while having a relative speed of around 1.66 meter/sec. Assuming perfect synchronization, we find that $|s(t)|^2$ and $|h(t)|^2$ approach $\delta(t)$ with the largest eigenvalue approaches to 1. This agrees with recent observations [4] that capacity-achieving signalings are peaky in nature.

3.1. Frequency Constraints

Obviously, δ functions are not practical signaling waveforms. One possible approach for designing practical signal/filter pairs is to introduce frequency restrictions to $s(t)$ and $h(t)$. Let $S(f) = \mathcal{F}(s(t))$ and $H(f) = \mathcal{F}(h(t))$ denote the Fourier transforms. We introduce the functional $\Omega_s(s)$ and $\Omega_h(h)$ as measures of the energy of $s(t)$ and $h(t)$ respectively in the frequency range $(-W, W)$,

$$\begin{aligned}\Omega_h(h) &= \int_{-W}^{W} H(f)H^*(f)\,df = \int[\Pi(\frac{f}{2W})H(f)]H^*(f)\,df\\ &= 2W\int\int \mathrm{sinc}(2\pi W(\tau-\lambda))h(\tau)h^*(\lambda)\,d\tau\,d\lambda,\end{aligned}$$

and similarly $\Omega_s(s) = 2W\int\int \mathrm{sinc}(2\pi W(\tau-\lambda))s(\tau)s^*(\lambda)\,d\tau\,d\lambda$.

We construct a Lagrangian to reflect the frequency containment of $h(t)$, i.e., $J(h) = (1-\beta)F(h) + \beta\Omega_h(h) - \gamma G(h)$ where $\beta \in [0,1]$ is a weighing factor. By setting the variation of $J(h)$ at h_0 to be zero, we obtain $(1-\beta)\frac{\delta}{\delta h}F(h_0) + \beta\frac{\delta}{\delta h}\Omega_h(h_0) = \gamma\frac{\delta}{\delta h}G(h_0)$. But,

$$\frac{\delta}{\delta h}\Omega_h(h_0) = 2W\int \mathrm{sinc}(2\pi W(\tau-\lambda))h_0^*(\lambda)\,d\lambda, \tag{4}$$

so, we have

$$\begin{aligned}\gamma h_0^*(\tau) = \int &[(1-\beta)R_\alpha(\tau,\lambda)\Psi(\tau,\lambda)\\ &+ 2\beta W\,\mathrm{sinc}(2\pi W(\tau-\lambda))]h_0^*(\lambda)\,d\lambda\end{aligned} \tag{5}$$

to characterize the optimal $h_0(t)$ for given $s_0(t)$. Similarly, we have

$$\begin{aligned}\gamma s_0^*(\tau) = \int &[(1-\beta)R_\alpha(\tau,\lambda)\Phi(\tau,\lambda)\\ &+ 2\beta W\,\mathrm{sinc}(2\pi W(\tau-\lambda))]s_0^*(\lambda)\,d\lambda,\end{aligned} \tag{6}$$

where $\Phi(\tau,\lambda) = \int h_0(\tau+\mu)h_0^*(\lambda+\mu)p(\mu)\,d\mu$, to characterize the optimal $s_0(t)$ for given $h_0(t)$.

Again, we can determine optimum signal/filter pairs with an iterative algorithm:

Algorithm 2: Initialize: Randomly select an initial random $s_0(t)$.

1. Set $h_0(t)$ to satisfy (5), where $\gamma = \gamma_1$ is the largest eigenvalue of the kernel and $h_0^*(t)$ is the corresponding eigenfunction.
2. Set $s_0(t)$ to satisfy (6) where $\gamma = \gamma_0$ is the largest eigenvalue of the kernel and $s_0(t)$ is the corresponding eigenfunction.
3. Repeat steps 1 and 2 until convergence.

An example computed using the same $R_\alpha(\tau)$ as before shows we are able to obtain signal/filter pair that are frequency constrained while at the same time matched to the channel autocorrelation and produce reasonable SNR. Fig. 2 shows the initial and final waveforms of $s(t)$ and $h(t)$, and how the largest eigenvalue increases with the iteration number. In this plot, $W = 2/\pi$Hz and $\beta = 0.6$.

4. SIGNAL SETS

To communicate over a fast fading channel, amplitude and phase modulation are inappropriate: something akin to orthogonal signaling is necessary. We must therefore find at least two sets of signal/filter pairs $(s_0(t), h_0(t))$ and $(s_1(t), h_1(t))$, such that under the channel influence $\alpha(t)$, $s_i(t)$ and $h_j(t)$ have a minimal correlation for $i \neq j$.

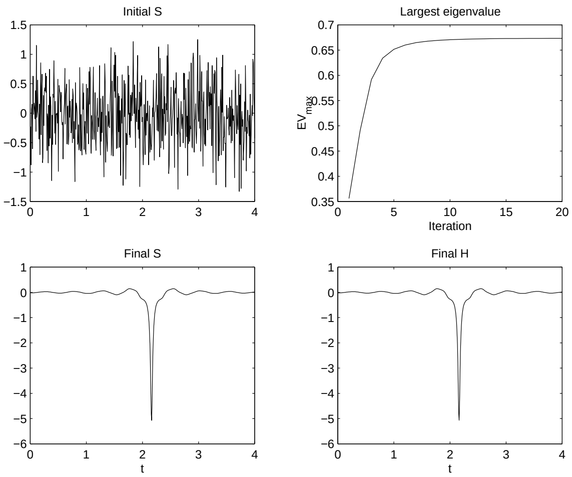

Fig. 2. Signal generation with frequency restrictions.

From (3), (5) and (6), we know that the minimum correlation can be obtain by setting $h_j(t)$ to be the eigenfunction corresponding to the smallest eigenvalue while we generate $h_i(t)$ for given $s_i(t)$, at the same time setting $s_j(t)$ to be the eigenfunction corresponding to the smallest eigenvalue while we generate $s_i(t)$ for given $h_i(t)$. However, the kernel must be changed to enforce frequency containment. So, while finding the eigenfunctions corresponding to the smallest eigenvalues, we use the kernel of the inverse operator ($\mathcal{I}[\cdot]$, Appendix 2) of that of (4) in place of $2W\,\mathrm{sinc}(2\pi W(\tau-\lambda))$ in (5) and (6) for frequency containment.

We suggest the following procedure: Given a transmitted signal $s_0(t)$ (resp. detection filter $h_0(t)$), let the corresponding filter $\hat{h}_0(t)$ (resp. signal $\hat{s}_0(t)$) be the eigenfunction having largest eigenvalue of the kernel $(1-\beta)R_\alpha(\tau,\lambda)\Psi(\tau,\lambda)+2\beta W\,\mathrm{sinc}(2\pi W(\tau-\lambda))$ (resp.$(1-\beta)R_\alpha(\tau,\lambda)\Phi(\tau,\lambda)+2\beta W\,\mathrm{sinc}(2\pi W(\tau-\lambda))$) as defined before. At the same time, set $\tilde{h}_1(t)$ (resp. $\tilde{s}_1(t)$) to be the eigenfunction with smallest eigenvalue of the kernel $(1-\beta)R_\alpha(\tau,\lambda)\Psi(\tau,\lambda)+\beta\mathcal{I}[2W\,\mathrm{sinc}(2\pi W(\tau-\lambda))]$ (resp.$(1-\beta)R_\alpha(\tau,\lambda)\Phi(\tau,\lambda)+\beta\mathcal{I}[2W\,\mathrm{sinc}(2\pi W(\tau-\lambda))]$). Use the same method to obtain $\hat{s}_1(t)$, $\hat{h}_1(t)$, $\tilde{s}_0(t)$, and $\tilde{h}_0(t)$. This procedure will be carried out repeatedly: for each new iteration, set $s_0(t)=\frac{\hat{s}_0(t)+\tilde{s}_0(t)}{2}$, $s_1(t)=\frac{\hat{s}_1(t)+\tilde{s}_1(t)}{2}$, $h_0(t)=\frac{\hat{h}_0(t)+\tilde{h}_0(t)}{2}$, and $h_1(t)=\frac{\hat{h}_1(t)+\tilde{h}_1(t)}{2}$, then normalize these functions to unit energy. The procedure is summarized in fig. 3. Computationally, the smallest eigenvalues can be found using inverse iteration. As before, the matrices require only vector-size storage.

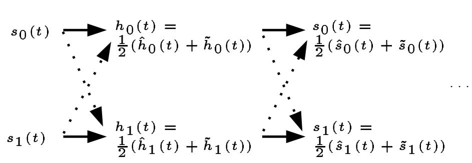

Fig. 3. Procedure for generating signal/filter pairs.

Due to the averaging in the generation process of the signal/filter pairs, the pairs vary a lot, both with the waveforms and the SNR in term of the largest eigenvalue. However, during the process, we come across many "good" pairs which have high SNR. Fig. 4 shows several of these pairs for a ranges of βs.

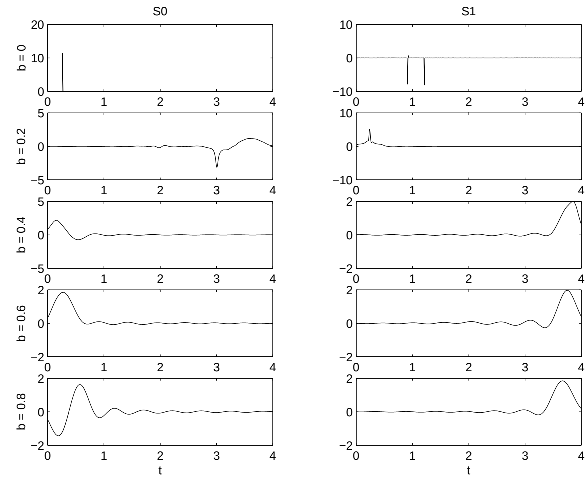

Fig. 4. Signal/filter pairs generated when $W=2/\pi$Hz.

A quadratic receiver as shown in fig. 5 is used to detect the received signal. Let $r_{k_i}=\int(\alpha(t)s_k(t)+n(t))h_i(t)dt$ denote

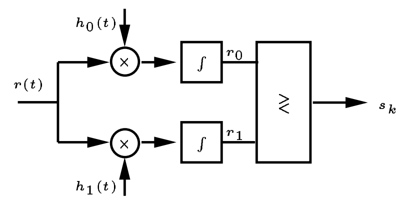

Fig. 5. Quadratic Receiver.

the signal-plus-noise receiver statistic when $s_k(t)$ is sent, and let $\mathbf{r}_k=[r_{k_0}\ \ r_{k_1}]^T$. Let $\bar{\mathbf{r}}_k=E(\mathbf{r}_k)$. The covariance matrix of the sample vector when $s_k(t)$ is transmitted is defined as $R_k=E[(\mathbf{r}_k-\bar{\mathbf{r}}_k)(\mathbf{r}_k-\bar{\mathbf{r}}_k)^H]$. Since correlation detection is a linear operation, $\mathbf{r}$ is complex Gaussian with density function [5]

$$p(\mathbf{r}|s_k(t))=\frac{1}{\pi^N|R_k|}\exp[-(\mathbf{r}-\bar{\mathbf{r}}_k)^H R_k^{-1}(\mathbf{r}-\bar{\mathbf{r}}_k)].$$

The decision rule we use is

$$p(\mathbf{r}|s_0(t))-p(\mathbf{r}|s_1(t))\underset{s_1}{\overset{s_0}{\gtrless}}0$$

$$\Rightarrow(\mathbf{r}-\bar{\mathbf{r}}_1)^H R_1^{-1}(\mathbf{r}-\bar{\mathbf{r}}_1)-(\mathbf{r}-\bar{\mathbf{r}}_0)^H R_0^{-1}(\mathbf{r}-\bar{\mathbf{r}}_0)\underset{s_1}{\overset{s_0}{\gtrless}}\ln\frac{|R_0|}{|R_1|}$$

$$\Rightarrow g=\mathbf{v}^H Q\mathbf{v}\underset{s_1}{\overset{s_0}{\gtrless}}\xi,$$

where $Q=R_1^{-1}-R_0^{-1}$, $\mathbf{w}=2(R_0^{-1}\bar{\mathbf{r}}_0-R_1^{-1}\bar{\mathbf{r}}_1)$, $\mathbf{u}=-\dfrac{1}{2}Q^{-1}\mathbf{w}$, $\mathbf{v}=\mathbf{r}-\mathbf{u}$, and $\xi=\ln\frac{|R_0|}{|R_1|}+\bar{\mathbf{r}}_1^H R_1^{-1}\bar{\mathbf{r}}_1-\bar{\mathbf{r}}_0^H R_0^{-1}\bar{\mathbf{r}}_0-\frac{1}{4}\mathbf{w}^H Q^{-1}\mathbf{w}$. The quadratic form in the exponent can be diagonalized by using the transformation $E(\mathbf{v})=\bar{\mathbf{v}}=U_1L^{-1}U_2\mathbf{d}$ where U_1 and U_2 are obtained from the unitary eigendecomposition $M^{-1} = U_1 L^2 U_1^H$ and $L^{-1} U_1^H Q U_1 L^{-1} = U_2 \Omega U_2^H$, and where Ω is a diagonal matrix of the eigenvalues of MQ. For Rayleigh fading, we have $\bar{\mathbf{v}} = 0$ and the probability of error rate then can be evaluated using the cumulative distribution function of g presented in [6], i.e.,

$$F_g(\xi) = \begin{cases} \sum_{\varrho_i<0} \left\{ \prod_{n=1,n\neq i}^{N} \frac{1}{1-\frac{\varrho_n}{\varrho_i}} \right\} \exp(-\xi \varrho_i^{-1}) & \text{if} \quad \xi < 0 \\ 1 - \sum_{\varrho_i>0} \left\{ \prod_{n=1,n\neq i}^{N} \frac{1}{1-\frac{\varrho_n}{\varrho_i}} \right\} \exp(-\xi \varrho_i^{-1}) & \text{if} \quad \xi \geq 0, \end{cases}$$

where ϱ_k are the eigenvalues of MQ.

Fig. 6 shows a performance comparison between the signal/filter pairs obtained by method suggested in this paper with binary flat-top pulses in a PPM pair with a span of 4 seconds through a typical land motion fading channel. In this plot, $SNR = -10 \log_{10} N_0/2$ (signals of constant unit energy).

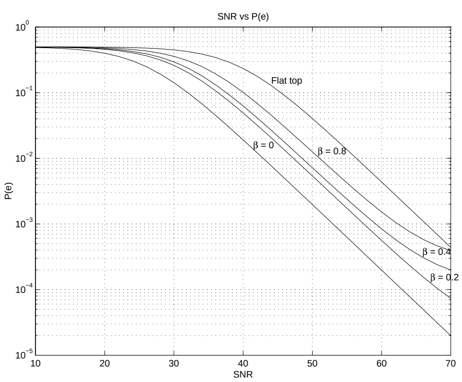

Fig. 6. $P(e)$ vs SNR.

5. CONCLUSION

We characterized the optimal detector and signaling in term of SNR for a fading channel when the autocorrelation function of the channel variation was known. Our result included the well known Matched filter and Rake receiver. We also provided an algorithm that generates optimal signal/filter pairs having good SNR which satisfy frequency containment. With typical land mobile fading channel, the signals obtained with our procedure have a much better performance compare to traditional flat-top pulse in term of probability of error rate.

6. APPENDIX 1

The derivative, more precisely the variation, of F: $\mathcal{L}_2(\mathbb{R}) \to \mathbb{C}$ with respect to h is a function defined as follows. Let $h(t) = u(t) + jv(t)$, where both $u(t)$ and $v(t)$ are real, we have [7] $\frac{\delta F(h)}{\delta h} = \frac{1}{2}\left(\frac{\delta F(h)}{\delta u} - j\frac{\delta F(h)}{\delta v}\right)$, and $\frac{dF(h)}{dh^*} = \frac{1}{2}\left(\frac{\delta F(h)}{\delta u} + j\frac{\delta F(h)}{\delta v}\right)$, such that for $\epsilon \in \mathbb{R}$, and $V \in \mathcal{L}_2(\mathbb{R})$[8],

$$\begin{aligned} \langle \frac{\delta}{\delta u} F(h)\Big|_{h_0}, V\rangle &= \langle \frac{\delta F(h_0)}{\delta u}, V\rangle = \int (\frac{\delta F(h_0)}{\delta u})(t) V(t) dt \\ &= \lim_{\epsilon \to 0} \frac{F((u_0 + \epsilon V) + jv_0) - F(h_0)}{\epsilon}, \end{aligned}$$

and

$$\langle \frac{\delta}{\delta v} F(h)\Big|_{h_0}, V\rangle = \lim_{\epsilon \to 0} \frac{F(u_0 + j(v_0 + \epsilon V)) - F(h_0)}{\epsilon}.$$

That is, the variation $\frac{\delta F}{\delta u}$ is that function which provides a linear approximation to F.

7. APPENDIX 2

Let $k(\lambda, \tau) = \mathcal{I}[g(\lambda, \tau)]$, so that we have $\int k(\lambda, \tau) g(\tau, \nu) d\tau = \delta(\lambda - \nu)$. For the case $k(\lambda, \tau) = k(\lambda - \tau)$ and $g(\lambda, \tau) = g(\lambda - \tau)$, we have

$$\begin{aligned} &\int k(\lambda - \tau) g(\tau - \nu) d\tau = \delta(\lambda - \nu) \\ \Rightarrow &\int k(t - u) g(u) du = \delta(t) \Rightarrow K(\omega) = \frac{1}{G(\omega)}, \end{aligned}$$

where $K(\omega)$ and $G(\omega)$ are the Fourier transforms of $k(t)$ and $g(t)$ respectively. If $G(\omega) = 0$ for some ω and is bounded below, we use $K(\omega) = \frac{1}{G(\omega) + I}$ as the approximation of the inverse operator kernel, with I being some constant such that $G(\omega) + I > 0$.

8. REFERENCES

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