NONPARAMETRIC ESTIMATION OF INTERACTION FUNCTIONS FOR TWO-TYPE PAIRWISE INTERACTION POINT PROCESSES

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ABSTRACT

Nonparametric estimation of interaction functions for two-type pairwise interaction point processes is addressed. Such a problem is known to be challenging due to the intractable normalizing constant present in the density function. It is shown that the means of the marked interpoint distance functions embedded in the two-type pairwise interaction point process converge to the means of an inhomogeneous Poisson processes. This suggests a simple and effective nonparametric estimation method. An example is presented to illustrate the efficacy of our method. Our results can be generalized to multitype point processes in a straightforward manner, although the notation is more involved.

1. INTRODUCTION

The class of *pairwise interaction point processes* has been used in applications such as cosmology, ecology, forestry, and seismology [4]. The density functions of these processes are completely characterized by a univariate *interaction function*. It is well known that estimation of the interaction function is nontrivial due to the normalizing constant in the density function. The problem of estimating this function for the case where only a single type of point is present has been addressed in several papers, e.g., [3], [5]–[7], [11]–[13].

In this paper, we focus on *multitype* pairwise interaction point processes, where there is more than one type of point. In particular, we consider estimating the interaction functions nonparametrically. Estimation of interaction functions for multitype pairwise interaction point processes has been considered in [1], [9], [8]. However, to the best of our knowledge, no work on nonparametric estimation of multitype pairwise interaction point processes has ever been published.

*The work of the second author was performed while he was with the University of Wisconsin–Madison.

This paper is organized as follows. In Section 2, the mathematical definition of two-type pairwise interaction point processes is given. We present a limit theorem regarding the means of the embedded interpoint distance functions in Section 3. This theorem leads to the nonparametric estimation method we propose in Section 4. A numerical example is given in Section 5. We conclude this paper in Section 6.

2. MATHEMATICAL MODEL

Let $\mathbb{M}$ be a fixed finite set; $\mathbb{M}$ will be the mark space. Without loss of generality, we let $\mathbb{M} = \{0, 1\}$ in this paper, i.e., we focus on two-type point processes. It is straightforward to extend our results to multitype cases, although the notation is more involved. For each $n = 2, 3, 4, \ldots$, let $\mathbb{D}_n$ be a bounded set in the plane $\mathbb{R}^2$ equipped with Euclidean norm $\|\cdot\|$. A $(\mathbb{D}_n \times \mathbb{M})^n$-valued random vector

$$\Xi_n = (X_n, M_n) = ((X_{1,n}, M_{1,n}), \ldots, (X_{n,n}, M_{n,n}))$$

is called a *two-type conditional binomial process* with relative intensity b if it has a density function of the form

$$\tilde{f}_n(\xi) \;=\; \frac{1}{\tilde{Z}_n} b^{n_1(m)},$$

where $\xi = (x, m) = ((x_1, m_1), \ldots, (x_n, m_n))$, $\tilde{Z}_n$ is the normalizing constant so that $\int_{\mathbb{D}_n \times \mathbb{M}} \tilde{f}_n(\xi)\, d\xi = 1$, $b > 0$ is called the relative mark intensity, and $n_1(m)$ is the number of type-1 points in ξ. The name comes from the fact that, conditioned on $n_1(M)$, the mark-1 points form a binomial process and so do the mark-0 points. Furthermore, the $M_{i,n}$ are Bernoulli random variables with the probability of a success b times larger than that of a failure.

A *two-type pairwise interaction point process* with relative mark intensity b has a density function f_n with respect to the distribution of the two-type conditional binomial process with the same relative mark intensity, where

$$f_n(\xi) \;=\; \frac{1}{Z_n} \prod_{1 \le i < j \le n} \varphi_{m_i, m_j}(\|x_i - x_j\|). \qquad (1)$$

In (1), the $\varphi_{v,w}$ are called *interaction functions* with $\varphi_{v,w} = \varphi_{w,v}$ for all $v, w \in \mathbb{M}$, and Z_n is the normalizing constant. The idea is that if $\varphi_{v,w}(r) < 1$, then realizations in which many point pairs with $||x_i - x_j|| \approx r$ and $(m_i, m_j) = (v, w)$ will have low probability. Since $\varphi_{v,w} = \varphi_{w,v}$ and since $\mathbb{M} = \{0, 1\}$, we can write $\varphi_{v,w} = \varphi_{v+w}$. Hence,

$$f_n(\xi) \;=\; \frac{1}{Z_n} \prod_{1 \le i < j \le n} \varphi_{m_i+m_j}(||x_i - x_j||).$$

Note that Z_n is computationally intractable except for trivial φ_{v+w}.

In this paper, we restrict our attention to processes showing inhibition, i.e., $0 \le \varphi_{v+w} \le 1$, for all $v, w \in \mathbb{M}$. In addition, we assume that the interaction functions all have finite interaction ranges. That is, there exists $R > 0$ such that $\varphi_{v+w}(r) = 1$ for $r > R$ and for $v, w \in \mathbb{M}$. Hence, realizations in which pairs of points that are close are discouraged, while pairs of points that are far from each other are neither encouraged nor discouraged.

3. LIMIT THEOREM

In this section, we establish a limit theorem regarding the means of the *marked interpoint distance functions* $S_\xi^k(\cdot)$ embedded in the two-type pairwise interaction point process, where

$$S_\xi^k(r) \;:=\; \sum_{1 \le i < j \le n} \mathbf{1}_{\{k\}}(m_i + m_j)\mathbf{1}_{(0,r]}(||x_i - x_j||),$$

for $k = 0, 1, 2$. For each k and as a function of r, $S_\xi^k(r)$ is the number of point pairs with certain mark-pair combinations in the realization ξ with pairwise distance less than or equal to r. It is a generalization of the *interpoint distance function* [10] in the one-type case.

Our limit theorem is based on the *sparseness conditions* [14], which are purely geometrical requirements on how fast the region $\mathbb{D}_n$ grows relative to the number points n. The precise conditions are

$$\lim_{n\to\infty} \frac{n(n-1)}{2} \frac{\pi |I_n(r)|}{|\mathbb{D}_n|^2} \;=\; \lambda, \qquad \lambda > 0, \tag{2}$$

$$\lim_{n\to\infty} \frac{n(n-1)}{2} \int_{\mathbb{D}_n \setminus I_n(r)} \frac{|B_r(\eta) \cap \mathbb{D}_n|}{|\mathbb{D}_n|^2} \, d\eta \;=\; 0, \tag{3}$$

where $|\cdot|$ is the area function, $B_r(\eta)$ is the ball centered at η of radius r,

$$B_r(\eta) \;:=\; \{\eta' \in \mathbb{R}^2 : ||\eta - \eta'|| \le r\}, \qquad \eta \in \mathbb{R}^2,$$

and $I_n(r)$ is the set of η for which the ball $B_r(\eta) \subset \mathbb{D}_n$,

$$I_n(r) \;:=\; \{\eta \in \mathbb{D}_n :\; B_r(\eta) \subset \mathbb{D}_n\}, \qquad n = 2, 3, \ldots .$$

Theorem 1 *Denote by* $\hat{\mathsf{P}}_n$ *the distribution of the* n*-point two-type pairwise interaction point process* Ξ_n *with relative mark intensity* b *and with piecewise-continuous interaction functions* φ_i*. If the sparseness conditions* (2), (3) *are satisfied and if the* $\mathbb{D}_n$ *are such that* $1/|\mathbb{D}_n| = O(n^{-2})$*, then, for* $k = 0, 1, 2$,

$$\mathsf{E}_{\hat{\mathsf{P}}_n}[S_{\Xi_n}^k(r)] \;\to\; \hat{\Lambda}_k\big((0, r]\big), \qquad \text{as } n \to \infty,$$

where

$$\hat{\Lambda}_0(A) \;:=\; \frac{1}{(1+b)^2} \int_A 2\lambda r \varphi_0(r)\, dr,$$

$$\hat{\Lambda}_1(A) \;:=\; \frac{2b}{(1+b)^2} \int_A 2\lambda r \varphi_1(r)\, dr,$$

$$\hat{\Lambda}_2(A) \;:=\; \frac{b^2}{(1+b)^2} \int_A 2\lambda r \varphi_2(r)\, dr,$$

for A *a Borel subset of* $(0, R]$.

The proof is given in [2].

4. NONPARAMETRIC ESTIMATION

Consider an N-point (N is a fixed constant) two-type pairwise interaction point process corresponding to unknown inhibitive interaction functions φ_k, $k = 0, 1, 2$, with finite interaction ranges, in an observation region $\mathbb{D}$. Since

$$\frac{d}{dr} \int_0^r 2\lambda r' \varphi_k(r')\, dr' \;=\; 2\lambda r \varphi_k(r),$$

if N is large, then by Theorem 1, the φ_k can be approximated by

$$\hat{\varphi}_0(r) \;=\; \frac{(1+b)^2}{2\lambda r} \left(\frac{d\mathsf{E}_{\hat{\mathsf{P}}_N}[S_{\Xi_N}^0(r)]}{dr} \right), \tag{4}$$

$$\hat{\varphi}_1(r) \;=\; \frac{(1+b)^2}{4\lambda b r} \left(\frac{d\mathsf{E}_{\hat{\mathsf{P}}_N}[S_{\Xi_N}^1(r)]}{dr} \right), \tag{5}$$

$$\hat{\varphi}_2(r) \;=\; \frac{(1+b)^2}{2\lambda b^2 r} \left(\frac{d\mathsf{E}_{\hat{\mathsf{P}}_N}[S_{\Xi_N}^2(r)]}{dr} \right). \tag{6}$$

Suppose we observe independent realizations $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(J)}$, of the two-type pairwise interaction point process. We can estimate the expectations in (4)–(6) by the following sample means

$$\hat{\mathsf{E}}_{\hat{\mathsf{P}}_N}[S_{\Xi_N}^k(r)] \;:=\; \frac{1}{J} \sum_{j=1}^{J} S_{\xi^{(j)}}^k(r).$$

In order to compute the derivatives, we use cubic splines to fit the sample means first and then differentiate the approximations since differentiation of splines is straightforward.

Note one would still have to know the values of λ and b in order to use the estimators (4)–(6).

In [14], it was shown that $\mathsf{E}_{\mathsf{P}_n}[S_{X_n}(r)] \to \lambda r^2$, as $n \to \infty$, under the sparseness conditions, where P_n is the distribution of the *single*-type n-point binomial process. Hence, given that N is large, we can generate K realizations of the N-point binomial process $\{X_N^{(i)}\}_{i=1}^K$ by Monte–Carlo simulation and estimate λ by

$$\tilde{\lambda} \;=\; \frac{1}{Kr^2}\sum_{i=1}^{K} S_{X_N^{(i)}}(r), \tag{7}$$

Note that, since points in each realization $X_N^{(i)}$ of the binomial process are independent and identically uniformly distributed, the simulation is fairly easy.

Estimation of b is more difficult. However, we may circumvent it by the following procedure. First, define

$$\tilde{\varphi}_k(r) \;:=\; \frac{1}{2\lambda r}\left(\frac{d\hat{\mathsf{E}}_{\hat{\mathsf{P}}_N}[S^{i}_{\Xi_N}(r)]}{dr}\right), \qquad k = 0, 1, 2.$$

The $\tilde{\varphi}_k$ are scaled versions of the estimates $\hat{\varphi}_k$ with unknown scaling factors, say γ_k. In other words, $\tilde{\varphi}_k = \gamma_k \hat{\varphi}_k$. Once the γ_k are known, we can recover the $\hat{\varphi}_k$. Since the φ_k have finite interaction ranges, say R_k, $\varphi_k(r) = 1$, for $r \geq R_k$, $k = 0, 1, 2$. So ideally, $\tilde{\varphi}_k(r) = \gamma_k$, for $r \geq R_k$. But since the estimates $\hat{\mathsf{E}}_{\hat{\mathsf{P}}_N}[S^{i}_{\Xi_N}(r)]$ were used in computing the $\tilde{\varphi}_k$, the actual curves of $\tilde{\varphi}_k(r)$ should exhibit oscillations around the γ_k, for $r \geq R_k$, making it difficult not only to obtain the γ_k but also the interaction ranges R_k. In practice, one should plot the curves of the $\tilde{\varphi}_k$ and find the respective $\hat{R}_k$ beyond which the curves have leveled. By averaging the values of the $\tilde{\varphi}_k(r)$ for $r \geq \hat{R}_k$, one can obtain the estimates of the γ_k using these averages.

5. NUMERICAL EXAMPLE

In our numerical example, we consider the 200-point two-type point process in the 10-by-10 square with the following family of interaction functions [12]

$$\varphi_k(r) \;=\; 1 - e^{-\beta_k r^2}, \qquad k = 0, 1, 2,$$

with $(\beta_0, \beta_1, \beta_2) = (20, 32, 12)$. Although these interaction functions are completely parametrized by the β_k, we did not assume the parametrized forms in our estimation. The relative mark intensity was $b = 1.2$ and assumed unknown. We performed our estimation under two difference conditions. Under the first condition, our observations consisted of only 20 realizations of this point process, while under the second condition, we had 500 realizations as our observations.

To obtain $\tilde{\lambda}$ in (7), we generated 2500 samples of the 200-point single-type binomial process. The sample means $\hat{\mathsf{E}}_{\hat{\mathsf{P}}_N}[S^{k}_{\Xi_N}(r)]$ were computed at 80 equally spaced points in the interval $[0, 0.8]$. We plotted the estimates of the φ_k in Figs. 1–3. In these figures, the true interaction functions were plotted in solid lines, estimates obtained by using 20 realizations were in dotted lines, and estimates obtained by using 500 realizations were in dashed lines.

6. CONCLUSIONS

In this paper, we first established the theorem regarding the convergence of the means of the marked interpoint distance process embedded in the two-type pairwise interaction point process with piecewise-continuous interaction functions. This result can be generalized to the multitype case in a straightforward way, although the notation would be more complicated. Based on our limit theorem, we proposed a nonparametric estimator for the interaction functions. The results in our numerical example suggest that our methods work quite well.

We must emphasize that two-type pairwise interaction point processes are more complicated, as suggested by [8]. The complication arises from the more complex interaction structure; there are not only interactions between points of the same type, but also interactions between points of different types. In addition, with the same total number of points, information available per interaction is less in the two-type setting than that in the one-type case because there are more types of interactions. Therefore, results are less accurate in the two-type case if the total number of points remains the same. This may also suggest that the convergence to the limit is slower in the two-type setting.

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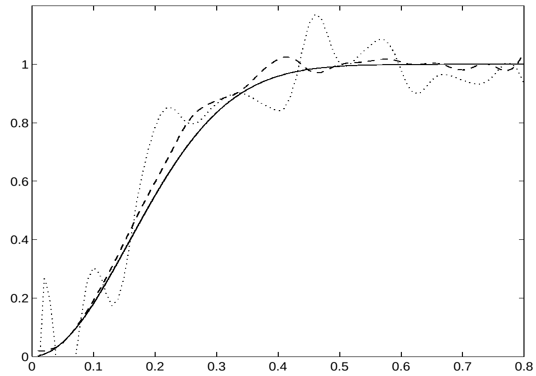

Fig. 1. Estimates of φ_0.

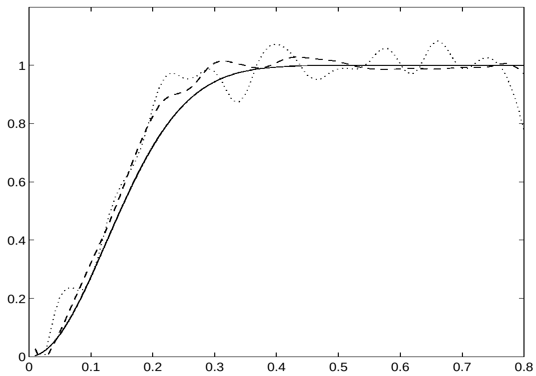

Fig. 2. Estimates of φ_1.

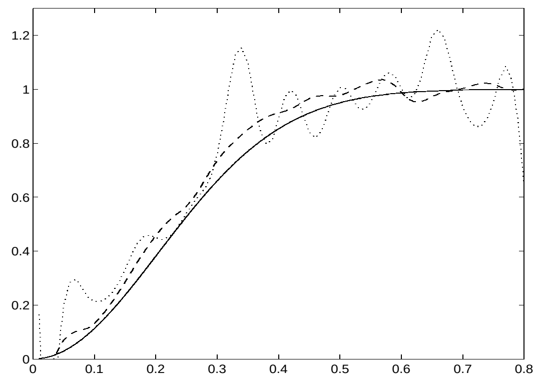

Fig. 3. Estimates of φ_2.

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