EFFECTING FREQUENCY SHIFTS WITH A DFT FILTERBANK

Gareth Parker †‡

† Defence Science and Technology Organisation
PO BOX 1500, Salisbury, South Australia
gareth.parker@dsto.defence.gov.au
‡ Institute for Telecommunications Research, University of South Australia

ABSTRACT

It is often required to apply a shift in frequency to the channelised data within a DFT filterbank. An example application is the frequency domain implementation of the cyclic Wiener filter. A common approach is to rotate the transform through an appropriate number of bins, but this is only accurate if the frequency shift is a multiple of the bin width. A better approach is to combine the bin rotation with an approximate fine shift. In this paper the exact solution is found for an arbitrary DFT filterbank and novel, computationally efficient approximations to this are derived and compared.

1. INTRODUCTION

The DFT filterbank [1], sometimes known as a transmultiplexer [2, 3] is a processing architecture commonly used to channelise time domain data into an approximate frequency domain. Once in the frequency domain it is often required to translate the data in frequency. This may be for the explicit purpose of receiver tuning or it may be required for some other operation such as cyclic Wiener filtering [4, 5] or cyclic spectral estimation [6, 7].

A frequency shift of $-\epsilon$ Hz may be applied to the discrete signal $x(n)$ by time domain mixing with the complex oscillator $\psi(n) = e^{-j2\pi\epsilon n/f_s}$, where f_s is the sampling frequency. We seek an operator $g\left\{\cdot\right\}$ to achieve this same result in the discrete frequency domain so that $g\left\{\mathcal{F}\left\{x(n)\right\}\right\} = \mathcal{F}\left\{e^{-j2\pi\epsilon n/f_s}x(n)\right\}$. We implement the transformation into the frequency domain, $\mathcal{F}\left\{\cdot\right\}$ using a K bin filterbank analyser [1] and the inverse transformation back to the time domain is achieved with a synthesis filterbank. Computationally efficient approximations to the frequency shift are also sought. For filtering applications, it is generally required to have both small mean square error and high coherence between the exact and approximately shifted frequency components.

The output of the analysis filterbank at time mM/f_s is a vector of bins $\mathbf{X}(m) = [X(m, f_0), \cdots, X(m, f_{K-1})]$, where M is the filterbank decimation rate. The kth bin contains an estimate of the complex envelope (sometimes called a 'complex demodulate' [6]) of the narrow bandpass filtered component of $x(n)$ centred at f_k Hz and is expressed as

$$X(m, f_k) = \sum_{n=-\infty}^{\infty} h(n)x(mM - n)e^{-j2\pi(mM-n)f_k/f_s}. \tag{1}$$

We consider the bins to be uniformly spaced between $-f_s/2$ and $f_s/2$ and so the bin width is equal to $f_b = f_s/K$. Using equation (1), $\mathcal{F}\left\{\psi(n)x(n)\right\}_k$, the kth bin of the transform of $x(n)$ shifted in the time domain by $-\epsilon$ Hz can be shown to be equal to $X(m, f_k + \epsilon)$, the frequency component of $x(n)$ centred at $f_k + \epsilon$. The vector of all such shifted bins corresponding to $k = 0 \ldots K-1$ is denoted $\mathbf{X}_\epsilon(m)$. If $\epsilon = \Delta f_b$, where Δ is an integer, the frequency shift is effected almost trivially using a circular rotation through the Δ filterbank bins [5]. However, this method is only exact when the required shift is a multiple of the bin width and in practice a residual fine shift will normally be required.

2. EXACT FINE FREQUENCY SHIFT

The analysis filter has an impulse response $h(n)$ with finite length $N_h = RK$ and the summation in equation (1) is performed over $T = N_h/f_s$ seconds. Using equation (1), taking an exponential term outside the summation and with a change of variables we can write

$$X(m, f_k + \epsilon) = e^{-j2\pi mMf_k/f_s} \sum_{r=0}^{N_h-1} h(N_h - r) \times \\ x(mM - N_h + r)\psi(mM - N_h + r)e^{-j2\pi rf_k/f_s}. \tag{2}$$

When $N_h = K$ the summation in (2) may be computed by direct application of a K point DFT. Define $\mathbf{E}(m) = \left[e^{j2\pi mMf_0/f_s}, \cdots, e^{j2\pi mMf_{K-1}/f_s}\right]$ and denote the DFT computed on the block $h(N_h - r)x(mM - N_h + r)$, $r = 0 \ldots K-1$ by $\text{DFT}\left\{h(N_h)x(mM - N_h)\right\}$. Using the vector middle product defined as $[a_1, \ldots, a_n] \otimes [b_1, \ldots, b_n] = [a_1b_1, \ldots, a_nb_n]$ we can write

$$\mathbf{X}(m) = \mathbf{E}^*(m) \otimes \text{DFT}\left\{h(N_h)x(mM - N_h)\right\} \tag{3}$$

$$\mathbf{X}_\epsilon(m) = \mathbf{E}^*(m) \otimes \text{DFT}\left\{h(N_h)x(mM-N_h)\psi(mM-N_h)\right\}. \quad (4)$$

Using the relation $\text{DFT}\{ab\} = 1/K\ \text{DFT}\{a\} * \text{DFT}\{b\}$ with $*$ denoting circular convolution and defining the DFT computed on $\psi(mM - N_h + r)$ by $\text{DFT}\{\psi(mM-N_h)\}$, equation (4) can be simplified. As $\psi(n)$ is a complex exponential, its DFT may be computed at time mM as its $m=0$ DFT, Ψ_0, mixed with a $-\epsilon$ Hz oscillator at the decimated sampling rate. Using this update and equation (3) achieves the aim of effecting an exact frequency shift of $x(n)$ by operating upon $\mathbf{X}(m)$,

$$\mathbf{X}_\epsilon(m) = \frac{1}{K} e^{-j2\pi(mM-N_h)\epsilon/f_s} \mathbf{E}^*(m) \otimes \left[(\mathbf{E}(m) \otimes \mathbf{X}(m)) * \Psi_0\right]. \quad (5)$$

The complexity is reduced if a simple approximation is made. As $\psi(n)$ is a complex exponential its DFT comprises only a few significant frequency samples and the circular convolution may be replaced with a truncated linear convolution with little loss in precision. In the simplest truncation, Ψ_0 may be replaced by single sample complex scalar $Ke^{j\phi}$ and (5) reduces further to

$$\mathbf{X}_\epsilon(m) \approx e^{-j2\pi mM\epsilon/f_s + j\phi}\mathbf{X}(m). \quad (6)$$

The frequency shift is approximated simply by mixing the filterbank subband components with a $-\epsilon$ Hz oscillator operating at the decimated sampling rate.

The more general case where $N_h = RK$, with $R > 1$ is particularly important and is preferred for most frequency domain filters [2]. Where $N_h = RK$, time aliasing the N_h length block of $h(N_h - r)\ x(mM - N_h + r)$ into a K sample block by stacking and adding may be used to manipulate the summation of (2) into a K point summation. The Weighted Overlap Add (WOLA) [1] filterbank architecture implements the summation as a double sum so that the filterbank output vector may be expressed as

$$\mathbf{X}(m) = \mathbf{E}^*(m) \otimes \sum_{l=0}^{R-1} \text{DFT}\left\{h(N_h - lK)x(mM - N_h + lK)\right\} \quad (7)$$

and similarly, the vector of all shifted demodulates is

$$\mathbf{X}_\epsilon(m) = \frac{1}{K} e^{-j2\pi(mM-N_h)\epsilon/f_s} \mathbf{E}^*(m) \otimes [\Psi_0 * \sum_{l=0}^{R-1} e^{-j2\pi\epsilon lK/f_s} \text{DFT}\left\{h(N_h - lK)x(mM - N_h + lK)\right\}]. \quad (8)$$

Comparison of equations (7) and (8) shows that in general, it is *not possible* to express $\mathbf{X}_\epsilon(m)$ in the form of a function of $\mathbf{X}(m)$.[1] However, two alternative methods of obtaining an exact shift are identified if the filterbank architecture is modified.

First, equation (8) may be implemented directly using a modified WOLA architecture. In the conventional WOLA architecture, for the mth data block, R segments of windowed data are stacked and added, with a K point DFT performed on the sum to generate $\mathbf{X}(m)$. An equivalent operation is to reverse the order of summation and perform a K point DFT on each segment, followed by a stack and add process. By multiplying each of the R DFTs by the appropriate exponential prior to summation, equation (8) is implemented.

A second alternative is to compute (2) using a $K' = N_h = RK$ point DFT and use the same approach as with $N_h = K$ to effect the fine frequency shift using equation (5). Following convolution, the K frequency domain samples are then obtained by decimating in frequency to retain only one in every R filterbank bins. The fine shifted filterbank bins may then be operated on as if they were directly generated using a K bin filterbank. A conventional K bin synthesis process can then be used.

The computational efficiency of these techniques can be increased by approximating the circular convolutions by truncated linear convolutions, as with the $R = 1$ processing.

3. SUBBAND MIXING

We have seen how for $R = 1$, a simple approximation to the exact shift is to mix the filterbank bins with a low sample rate, subband oscillator. This simple implementation may also be derived for arbitrary R, using a frequency-shift interpretation of the analysis filterbank. By appropriately grouping the exponential terms, equation (2) can be written as a filterbank transform using a modified analysis filter $h'(n) = h(n)e^{j2\pi\epsilon n/f_s}$, followed by a subband mixing with a $-\epsilon$ Hz oscillator. Approximate shift techniques can be found using suitable approximations for $h'(n)$. A crude approximation is $\tilde{h}(n) = h(n)e^{j\phi}$, which yields

$$\tilde{X}(m, f_k + \epsilon) \quad = \quad e^{-j2\pi\epsilon mM/f_s + j\phi} X(m, f_k) \quad (9)$$

This is the k^{th} bin of the filterbank transform, mixed in the transform domain with a $-\epsilon$ Hz oscillator. The frequency response of $X(f)$ and $\tilde{X}(f)$ are shown in Figure 1 for the case where $H(f)$ has an ideal bandpass form with bandwidth f_s/K. The frequency response of $\tilde{X}(f_k+\epsilon)$ only partially overlaps that of the desired $X(f_k + \epsilon)$. The term $\phi = \pi\epsilon(N_h - 1)/f_s$ ensures phase match in the region of spectral overlap. By instead using $\hat{h}(n) = h(n)(e^{j\phi} + e^{j2\pi n/K + j\phi_c})$

[1] unless the exponential multiplier within the summation of (8) reduces to a constant independent of l which will occur if ϵ is a multiple of the bin width f_s/K.

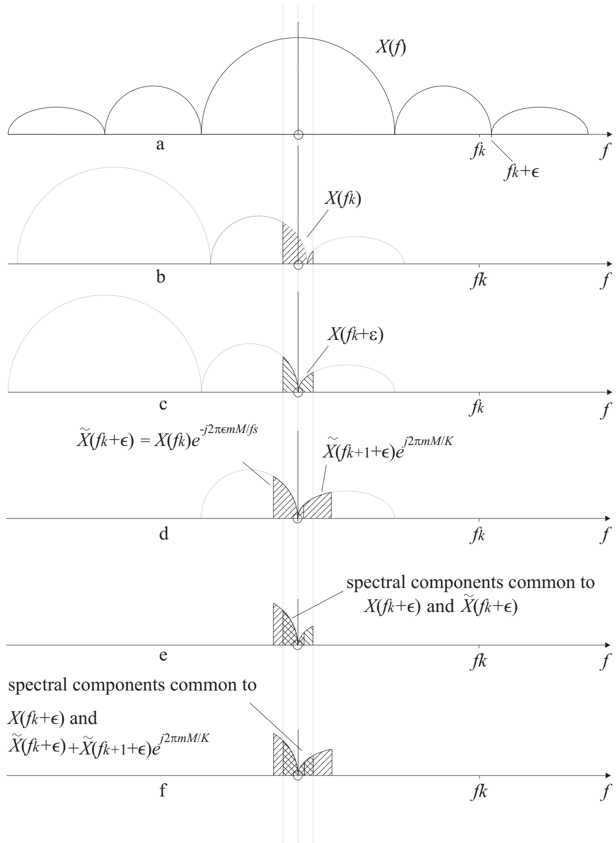

Fig. 1. Subband mixing and bin combining.

or equivalently, $\hat{H}(f) = H(f)e^{j\phi} + e^{j\phi_c}H(f - f_s/K)$, the amount of frequency overlap is increased, as seen in Figure 1. Again, ϕ_c is introduced to match the filter phase. By considering the expression for the analysis filterbank output using $\hat{h}(n) = h(n)(e^{j\phi} + e^{j2\pi n/K + j\phi_c})$, it can be shown [8] that

$$\hat{X}(m, f_k + \epsilon) = \tilde{X}(m, f_k + \epsilon) + e^{j2\pi mM/K + jR\pi}\tilde{X}(m, f_{k+1} + \epsilon). \quad (10)$$

The analysis filter extension is thus effected by performing a filterbank transform (using $h(n)$) on $x(n)$, fine frequency shifting using subband oscillator mixing and combining adjacent bins. A similar technique has been used for the special case of cyclic spectrum estimation [7].

4. TECHNIQUE COMPARISON

We undertook simulations to evaluate the accuracy of a baud rate frequency shift applied to an ideal baseband 16 kbaud BPSK signal. The SNR was computed as the mean ratio of the subband power in the exactly shifted signal to the error power in the approximate shift. The cross-coherence, ρ between the exact and approximate shifts was also computed.

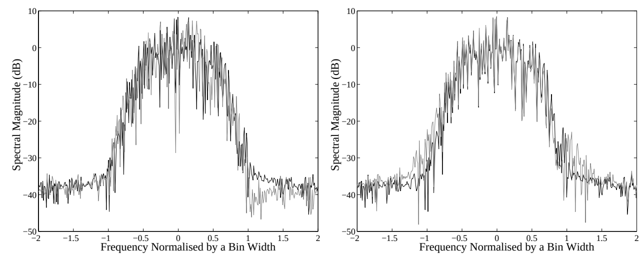

Fig. 2. Subband spectra for bin rotation (grey line left hand plot) and 3-point high resolution linear convolution (grey line right hand plot).

The results are listed in Table 1 using $K = 512$, $M = 128$ and $R = 3$. Interpretation of the results is made more clear by considering the sources of error. Figures 2 to 4 show the spectra of the shifted signal within a particular filterbank bin, for each of the approximate shifts (grey traces), overlaid with the spectra for the exact shift (black traces). The left hand plot of Figure 2 depicts a shift using bin rotation only. Although the approximate and exact shifts occupy the same bandwidth, the two correlate very poorly, as can be seen by the lack of fine structure commonality. In contrast, the right hand plot corresponds to the high resolution shift using a 3-point linear convolution. The exact and approximate shifts again occupy the same bandwidth, but this time there is a high degree of fine structure overlap, explaining the extremely high correlation. Figure 3 shows why a high coherence was also measured for the modified WOLA shift using a truncated 3-point convolution. However, the amount of noise outside the bandwidth of the bin suggests why the SNR for this shift is not as good.

The left and right hand plots of Figure 4 show the spectra for the shifts using subband mixing and bin combining respectively. While some correlation is suggested by the fine structure overlap of the left plot, it is clear why the correlation is inferior to the high resolution and modified WOLA shifts, since only a small fraction of the subband frequency components correlate. The coherence is increased by combing the adjacent bin, as seen in the right hand plot, where the exact and approximate shifts correlate over the entire bandwidth of the bin. However, the increased amount of out-of-band noise explains the relatively poor SNR.

5. DISCUSSION AND CONCLUSION

The test results clearly show that when it is feasible to use the high resolution convolution technique, it should be employed. It does however require the filterbank architecture to allow the analyser to operate with higher resolution than the synthesiser. The modified WOLA convolution approximation is more computationally efficient than the high res-

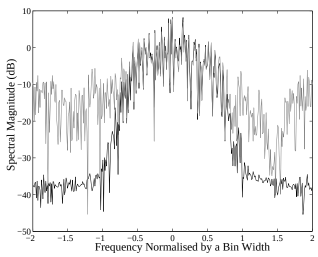

Fig. 3. Subband spectra for 3-point modified WOLA linear convolution.

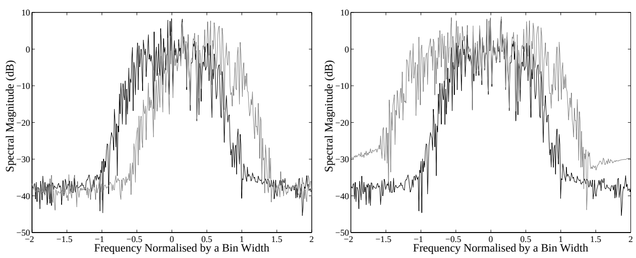

Fig. 4. Subband spectra for subband mixing (grey line left hand plot) and bin combining (grey line right hand plot).

olution approach, although the performance is not as good when few points are used in the convolution. This method requires the filterbank to be implemented using the modified WOLA architecture. The subband mixing approach is particularly simple and computationally efficient and generally provides quite reasonable results. However, when a relatively large fine shift is required, the coherence can be considerably increased with minimal additional complexity, by combining adjacent bins.

It should finally be noted that by suitable design of the synthesis filter, the out-of-band noise is reduced in the synthesis process. For frequency domain filtering applications this means a low subband SNR does not necessarily imply poor reconstruction performance. This issue is further addressed in [8].

Filters	ρ	**SNR (dB)**
circular convolution	1.00	> 100
high res. convolution (3 pts)	1.00	21.60
high res. convolution (1 pt)	0.99	17.59
WOLA convolution (3 pts)	0.93	10.31
WOLA convolution (1 pt)	0.77	4.54
bin combining	0.86	1.57
subband mixing	0.77	3.40
bin rotation only	0.08	-2.98

Table 1. Test SNR and Coherence.

6. REFERENCES

[1] R.E. Crochiere and L.R. Rabiner, *Multirate Digital Signal Processing*, Prentice-Hall, 1983.

[2] E.R. Ferrara, "Frequency-domain adaptive filtering," in *Adaptive Filtering*, Cowan and Grant, Eds. Prentice-Hall, 1985.

[3] G.C. Copeland, "Transmultiplexers used as adaptive frequency sampling filters," in *Proceedings of the IEEE International Conference on Acoustics Speech and Signal Processing*, 1982, pp. 319–322.

[4] W.A. Gardner, "Cyclic weiner filtering: Theory and method," *IEEE Transactions on Communications*, vol. 41, no. 1, pp. 151–163, January 1993.

[5] E.R. Ferrara, "Frequency-domain implementations of periodically time-varying filters," *IEEE Transactions on Acoustics Speech and Signal Processing*, vol. ASSP-33, no. 4, pp. 883–892, August 1985.

[6] W.A. Gardner, *Statistical Spectral Analysis : A Nonprobabalistic Theory*, Prentice-Hall, 1988.

[7] W.A. Brown and H.H. Loomis, "Digital implementations of spectral correlation analyzers," *IEEE Transactions on Signal Processing*, vol. 41, no. 2, pp. 703–720, February 1993.

[8] G.J. Parker, Ph.D. thesis, University of South Australia, to be submitted 2001.

7. ACKNOWLEDGEMENTS

The authors thanks Ken Lever, John Tsimbinos and Lang White for their helpful discussions relating to this work. The work was carried out with partial funding by the Cooperative Research Centre for Satellite Systems.