PIC-PROJECTION TECHNIQUE FOR DECREASING OF STATISTICAL ERROR IN SIGNALS RECONSTRUCTION PROBLEMS.

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ABSTRACT

As is well known, the problem of signal reconstruction may be reduced to estimation of coefficients of some signal decomposition. In the previous paper [1] we have shown, that optimal coordinate basis for this method must be formed from eigenfunctions of the Fisher's information operator (so-called PIC-basis).

However direct usage of the PIC-basis is not always convenient. Therefore in the given paper we propose the combined technique, according to which any given basis (FFT, DCT, wavelet, filterbanks, etc) is replaced by its projection on the subspace generated by the PIC-basis. Such projective basis, keeping all advantages of the initial representation, allows to decrease a range of possible fluctuations of a signal estimate.

The effectiveness of the proposed technique is illustrated by numerical examples from area of nonlinear tomography of a medium.

1. INTRODUCTION

In the modern signal processing various decompositions of the form

$$\mathbf{q} \;=\; \sum_{k=1}^{N} \gamma_k \, \varphi_k \;, \tag{1}$$

are widely used. Here $\mathbf{q}$ – a signal vector in some functional space, $\{\varphi_k\}_1^N$ – system of linearly independent basis vectors, γ_k - scalar coefficients of expansion.

In the given work a signal vector $\mathbf{q}$ is any useful information, extraction of which is the task of signal processing system (e.g. message, speech signal, some image, properties of some medium etc.). The essence of the methods, considered in the given paper, does not depend on a physical nature of particular tasks. However, with the purpose of clearness and simplicity of the consideration we assume, that the vector $\mathbf{q}$ belongs to some $M{-}D$ euclidean space[1], but the proposed methods without any difficulties can be extended to more complicated signal spaces.

Let's write expression (1) in compact vector-matrix notations:

$$\mathbf{q} \;=\; \boldsymbol{\Phi}\,\boldsymbol{\gamma} \tag{2}$$

where $\boldsymbol{\Phi}$ – basis matrix $M \times N$, which columns are vectors φ_k, $\boldsymbol{\gamma}$ – $N{-}D$ vector of expansion coefficients.

[1]At that, naturally, dimension of a basis $N \leq M$.

As for to the choice of a basis $\boldsymbol{\Phi}$ the theory of signal processing gives very wide opportunities: it is the basis of FFT, DCT, wavelet, filterbanks and many others [2]. Each of these bases has particular advantages and own application. So, in mathematical statistics the basis of the Principal Components (PC) is widely applied. This basis is formed from eigenfunctions of a signal correlation operator $\boldsymbol{\mathcal{R}}_{\mathrm{q}} \;=\; \langle\, \mathbf{q}\,\mathbf{q}^T \rangle$, where $\langle\cdot\rangle$ – symbol of the statistical averaging in a signal space. PC-basis of the given dimension N provides, as is well known [3], least mean square error of a signal representation.

However, the known approaches to the building of signal decomposition do not take into account specific character of the reconstruction problems, in which a signal is unavailable for direct observation. In such problems a vector of coefficients $\boldsymbol{\gamma}$ in the expansions (2) always is the result of a statistical estimation with using of noisy observations, which are connected with a signal by an arbitrary dependency

$$\mathbf{u} \;=\; \boldsymbol{\mathcal{H}}\,\{\,\mathbf{q}\,,\mathbf{n}\,\} \;=\; \boldsymbol{\mathcal{H}}\,\{\,\boldsymbol{\Phi}\,\boldsymbol{\gamma}\,,\mathbf{n}\,\}\,, \tag{3}$$

where $\mathbf{n}$ – noise vector, $\boldsymbol{\mathcal{H}}\{\cdot\}$ – some operator.

From this position in [4] the essentially new approach to the choice of optimum basis have been proposed. In accordance with this approach the Principal Components are those, which can be measured with a minimum error. At that the operator generating an optimum basis $\boldsymbol{\Phi}$, is not correlation operator of a signal as in classical method PC: it is well-known in the statistical estimation theory Fisher's information operator [5].

For the first time the useful properties of eigenfunctions of the Fisher's operator were ascertained in the paper [6]. However, the theoretical substantiation of the method optimality was not made. Such substantiation have been done in the paper [4]. Also the concept of informative components have been proposed. To distinguish the given interpretation of PC concept from the traditional viewpoint in [1] such components have been called as Principal Informative Components (PIC). This terminology also will be used in the given paper.

The further development of the PIC-method ideas was given in the work [1], where this method was extended both to the reconstruction problems of determined and random images. The given paper is continuation of the work [1].

2. PIC-METHOD

Let's consider the essence of the PIC-method for simple, but practically very important example, when the vector of observations $\mathbf{u}$ is connected with reconstructed signal $\mathbf{q}$ by the linear dependence:

$$\mathbf{u} = \mathbf{H}\mathbf{q} + \mathbf{n} = \mathbf{H}\boldsymbol{\Phi}\boldsymbol{\gamma} + \mathbf{n}, \tag{4}$$

where $\mathbf{H}$ is some matrix $P \times M$.

Let's assume, that noise is Gaussian with zero mean and correlation matrix $\boldsymbol{\mathcal{R}}_{\mathbf{n}}$. It is not difficult to show, that the maximum likelihood estimate for coefficients of a signal decomposition has the form:

$$\hat{\boldsymbol{\gamma}} = (\boldsymbol{\Phi}^T \mathbf{I}_q \boldsymbol{\Phi})^{-1} \boldsymbol{\Phi}^T \mathbf{H}^T \boldsymbol{\mathcal{R}}_{\mathbf{n}}^{-1} \mathbf{u}, \tag{5}$$

where

$$\mathbf{I}_q = \mathbf{H}^T \boldsymbol{\mathcal{R}}_{\mathbf{n}}^{-1} \mathbf{H} \tag{6}$$

is matrix $M \times M$. As was shown in [1], matrix form (6) is the Fisher's information matrix [5] for the problem of vector $\mathbf{q}$ estimation.

Using expression (5) for the estimate of a signal at noise background it is possible to write:

$$\hat{\mathbf{q}} = \boldsymbol{\Phi} (\boldsymbol{\Phi}^T \mathbf{I}_q \boldsymbol{\Phi})^{-1} \boldsymbol{\Phi}^T \mathbf{H}^T \boldsymbol{\mathcal{R}}_{\mathbf{n}}^{-1} \mathbf{u}. \tag{7}$$

Let's consider properties of the estimate (7). If the basis dimension $N < M$, the given estimate has bias, vector of which is equal to

$$\mathbf{b}_q = \mathbf{q} - \langle \hat{\mathbf{q}} \rangle = (\mathbf{E} - \boldsymbol{\Phi} (\boldsymbol{\Phi}^T \mathbf{I}_q \boldsymbol{\Phi})^{-1} \boldsymbol{\Phi}^T \mathbf{I}_q)\mathbf{q}, \tag{8}$$

where $\mathbf{E}$ is identity matrix.

Variance of the estimate (7) is defined by the expression

$$\sigma_q^2 = \langle | \hat{\mathbf{q}} - \langle \hat{\mathbf{q}} \rangle |^2 \rangle = \mathrm{Tr}\, \{\boldsymbol{\Phi}^T \boldsymbol{\Phi} (\boldsymbol{\Phi}^T \mathbf{I}_q \boldsymbol{\Phi})^{-1}\}. \tag{9}$$

It is not difficult to show, that stationary points of the functional (9) are reached on the set of eigenfunctions of the problem

$$\mathbf{I}_q \boldsymbol{\Phi} = \boldsymbol{\Phi} \boldsymbol{\Lambda}, \tag{10}$$

where $\boldsymbol{\Lambda}$ is diagonal matrix of N eigenvalues $\lambda_1, \ldots, \lambda_N$ for the operator $\mathbf{I}_q$. The solutions set of the problem (10), which satisfy the formulated below selection criteria, is called in [4] as the Principal Informative Components (PIC).

First selection criterion (shortly, the Criterion 1) includes in the PIC-basis those eigenvectors of the Fisher's matrix, which correspond to maximum eigenvalues[2] $\lambda_1, \ldots, \lambda_N$. It is not difficult to see, that such rule of the PIC-basis forming provides minimum of the statistical error (9) at the given basis dimension N

$$(\sigma_q^2)_{stat.} = \mathrm{Tr}\, \{\boldsymbol{\Lambda}^{-1}\} = \sum_{k=1}^{N} \frac{1}{\lambda_k}. \tag{11}$$

[2]It is supposed, that the eigenvalues are arranged in non-increasing order

Another criterion for the PIC-selection (shortly the Criterion 2) can be obtained, if within the framework of possible solutions of the problem (10) one puts forward the requirement of the total estimation error minimum, i.e.

$$d_q^2 = \langle | \hat{\mathbf{q}} - \mathbf{q} |^2 \rangle = | \mathbf{b}_q |^2 + \sigma_q^2 \rightarrow \min. \tag{12}$$

If basis $\boldsymbol{\Phi}$ is a solution of the equation (10), the total error (12) may be represented as:

$$d_q^2 = | \mathbf{q} |^2 - \sum_{k=1}^{N} (\gamma_k^2 - \frac{1}{\lambda_k}). \tag{13}$$

where γ_k – true coefficients of a signal decomposition.

From reviewing of the given expression it is possible to conclude, that the total estimation error is diminished only by such components, for which condition

$$\gamma_k^2 - \frac{1}{\lambda_k} \geq 0. \tag{14}$$

is satisfied. Total number of the PIC-components N_0, for which the condition (14) is satisfied, defines the optimum dimension of the basis $\boldsymbol{\Phi}$ by the Criterion 2.

Presented in the given section results refer to linear model of observations (4). In [1] it was shown, that the same methodology can be applied for the general model (3). The distinction is one that in general case it is possible to speak about an optimality of the PIC-basis with reference only to low bounds of estimation errors.

3. PIC-PROJECTION TECHNIQUE

A direct practical application of the PIC-basis is not always convenient, because the dimension of signal space M may be very large (hundreds and thousands). At reliable observations (SNR $\gg 1$) a total number of PIC can make up a considerable part from M. Other weakness of the PIC-basis - the lack of connection with a physical content of applied tasks.

At the same time, the decompositions used now in signal processing (FFT, DCT, wavelet, filterbanks and others) are more compact and more "physical", but are constructed without taking into account a statistical nature of signal reconstruction problems on noisy data. In the given section the technique is proposed which combines advantages of traditional physical representation with advantages of the PIC- method.

Let's assume, that the signal $\mathbf{q}$ is represented as a decomposition

$$\mathbf{q} = \boldsymbol{\Psi}\boldsymbol{\beta} \tag{15}$$

where $\boldsymbol{\Psi}$ – matrix $M \times Q$, the columns of which are some coordinate vectors $\boldsymbol{\psi}_k$, $(k = 1, \ldots, Q)$, $\boldsymbol{\beta}$ – Q–D vector of expansion coefficients.

Let's suppose that basis $\boldsymbol{\Psi}$ ensures compact representation of a clean signal, i.e. it is possible to neglect the systematic error of representation (15) already at rather small Q.

Let's represent the vector (15) on some other basis $\boldsymbol{\Phi}$ with dimension $N \geq Q$. Vector of expansion coefficients

$\boldsymbol{\gamma}$, selected from the condition $\| \mathbf{q} - \boldsymbol{\Phi}\, \boldsymbol{\gamma} \|_2 \;\rightarrow\;$ min, looks as:

$$\boldsymbol{\gamma} \;=\; (\boldsymbol{\Phi}^T \boldsymbol{\Phi})^{-1} \boldsymbol{\Phi}^T \boldsymbol{\Psi} \boldsymbol{\beta}\,.$$

Structure of the given expression allows to seek an estimate $\hat{\mathbf{q}}$ of a signal $\mathbf{q}$ in the form

$$\hat{\mathbf{q}} \;=\; \boldsymbol{\Psi}_\phi \, \hat{\boldsymbol{\beta}}\,, \tag{16}$$

where $\hat{\boldsymbol{\beta}}$ – an estimate of expansion coefficients for initial decomposition (15),

$$\boldsymbol{\Psi}_\phi \;=\; \boldsymbol{\Pi}_\phi \; \boldsymbol{\Psi}\;, \tag{17}$$

– some new basis[3] of dimension Q,

$$\boldsymbol{\Pi}_\phi \;=\; \boldsymbol{\Phi}(\, \boldsymbol{\Phi}^T \boldsymbol{\Phi}\,)^{-1} \boldsymbol{\Phi}^T \tag{18}$$

– operator of orthogonal projection on a subspace generated by a basis $\boldsymbol{\Phi}$.

The expression (16) is correct for arbitrary bases $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$. Now let's assume, that basis $\boldsymbol{\Phi}$ is the PIC-basis, i.e. is composed from solutions of the problem (10). Besides let's suppose, that PIC are selected by Criterion 1, i.e. in the decreasing order of eigenvalues $\lambda_1, \ldots, \lambda_N$. It is not difficult to show, that in this case the estimate (16) ensures a smaller statistical error of a signal reconstruction, than estimate of the form

$$\hat{\mathbf{q}} \;=\; \boldsymbol{\Psi} \, \hat{\boldsymbol{\beta}}\,, \tag{19}$$

i.e. estimate obtained with direct usage of the initial basis $\boldsymbol{\Psi}$.

Really, statistical error of the signal estimate (19) is defined by expression similar to (9) which is written for the basis $\boldsymbol{\Psi}$:

$$(\,\sigma_q^2\,)_{(19)} \;=\; \mathrm{Tr}\,\{\boldsymbol{\Psi}^T \boldsymbol{\Psi} (\boldsymbol{\Psi}^T \mathbf{I}_q \boldsymbol{\Psi})^{-1}\}. \tag{20}$$

The boundaries of the error (20) for an arbitrary basis $\boldsymbol{\Psi}$ can be estimated using the Courant-Fischer's theorem [7]:

$$\sum_{k=1}^{Q} \frac{1}{\lambda_k} \;\leq\; (\,\sigma_q^2\,)_{(19)} \;\leq\; \sum_{k=M-Q+1}^{M} \frac{1}{\lambda_k}\,. \tag{21}$$

Now let's consider the boundaries of the statistical error for the signal estimate (16). The given error is obviously equal to

$$(\,\sigma_q^2\,)_{(16)} \;=\; \mathrm{Tr}\,\{\boldsymbol{\Psi}^T \boldsymbol{\Pi}_\phi\, \boldsymbol{\Psi}\, (\boldsymbol{\Psi}^T \boldsymbol{\Pi}_\phi\, \mathbf{I}_q\, \boldsymbol{\Pi}_\phi\, \boldsymbol{\Psi}\,)^{-1}\}\,.$$

In view of structure of the projective operator (18) and equation (10) and taking into account orthonormal properties of the PIC-basis the given expression can be transformed to the form

$$(\,\sigma_q^2\,)_{(16)} \;=\; \mathrm{Tr}\,\{\boldsymbol{\Theta}^T \boldsymbol{\Theta} (\boldsymbol{\Theta}^T \boldsymbol{\Lambda} \boldsymbol{\Theta})^{-1}\}\,, \tag{22}$$

where $\boldsymbol{\Theta} \;=\; \boldsymbol{\Phi}^T\, \boldsymbol{\Psi}$, and $\boldsymbol{\Lambda}$, as above, diagonal matrix of higher eigenvalues of the information operator $\mathbf{I}_q$.

It is not difficult to see, that the expression (22) is similar by structure to expression (20), but instead of the Fisher's matrix with dimension M in this case we have the diagonal matrix $\boldsymbol{\Lambda}$ with dimension N. Therefore, by analogy with (21) the error (22) can be estimated by the inequality

$$\sum_{k=1}^{Q} \frac{1}{\lambda_k} \;\leq\; (\,\sigma_q^2\,)_{(16)} \;\leq\; \sum_{k=N-Q+1}^{N} \frac{1}{\lambda_k}\,. \tag{23}$$

From comparison of the expressions (21) and (23) it is easy to see, that under $N < M$ the range of possible values of the statistical error (23) is more narrow and the upper bound of this range is lower, than for the error (21). Therefore, a signal estimate of the form (16), obtained with usage of the projective basis (17), are less subjected to statistical errors, than estimate (19), which directly uses a physical basis.

It is possible to say, that the projector (18) acts on basis $\boldsymbol{\Psi}$ as a filter suppressing those components, which actually can not be restored in view of large statistical errors of measurements. Thus in a primary physical representation it is possible to take into account a statistical nature of the estimation procedure. The proposed projective approach has advantages also in comparison with the PIC-basis in the "pure" form. At first, the dimension of the projective basis $\boldsymbol{\Psi}_\phi$ is defined by dimension of the primary basis $\boldsymbol{\Psi}$ and is not too large, secondly, such representation is more physical.

4. NUMERICAL RESULTS

In the given section we shall consider the numerical example from the area of acoustic tomography of ocean medium [8], [9]. One of the major problems of the ocean acoustic tomography is the task of restoring of so-called Sound Speed Profile (SSP), i.e. the dependency of sound speed from depth: $c \;=\; c_0(z) \;+\; q(z)$, where $c_0(z)$ – statistical-average SSP, $q(z)$ – perturbation of the SSP, which is subjected to the estimation. Thus in this problem the components of the vector $(\,\mathbf{q}\,)_m \;=\; q(z_m)\,,\; m \;=\; 1, \ldots, M$ are the aim of estimation.

The following scheme of measurements is usually applied for solution of such problems. In some point of the ocean medium the source of acoustic signal is placed. The reception of a signal is carried out at some distance from the source by array of acoustic receivers. The spectrum of acoustic field $s(\omega, \mathbf{r})$ in the area of observation is connected with parameters of medium by the Helmholtz equation [10]. It is necessary to restore a vector $\mathbf{q}$ from observations of the field $u(\omega, \mathbf{r}) \;=\; s(\omega, \mathbf{r}) \;+\; n(\omega, \mathbf{r})\,.$

For perturbed part of SSP the model of paper [9] was accepted:

$$q(z) \;=\; \sum_{k=0}^{Q-1} \beta_k \;\vartheta_k(z)\,, \tag{24}$$

where $\vartheta_k(z)$ – so-called baroclinic modes of Rossby wave, which describe the perturbation of SSP as a result of interaction of tidal processes in ocean with the Earth rotation (first 3 modes are shown at fig.1),

$$\beta_k \;=\; 200\,\frac{(-1)^k}{2^k} \tag{25}$$

[3] By virtue of the condition $N \;\geq\; Q$ the coordinate vectors of the basis (17) are linearly independent [7].

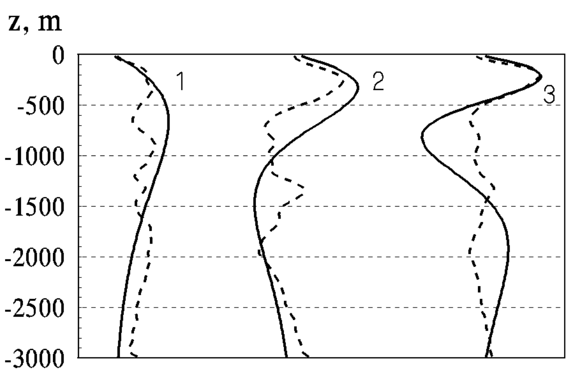

Figure 1: First 3 baroclinic modes od the SSP (dotted line - modes projection on the PIC-subspace).

– coefficients of modes excitation, Q – number of baroclinic modes ($Q = 10$).

Thus, in considered example the existence of physical basis for the reconstructed signal is initially supposed. Therefore, there is a possibility for effectiveness comparison of this physical basis and the PIC-basis.

The results of such comparison are displayed on fig.2. Here are shown such errors of the SSP estimation (m/s): total error $\sqrt{|\mathbf{b}_q|^2 + \sigma_q^2}/M$ (the curve 1), systematic error $|\mathbf{b}_q|/M$ (the curve 2) and statistical error σ_q/M (the curve 3) in dependence on the dimension of the physical (fig.1a) and projective (fig.1b) basis. It is easy to see, that in the given example the projective basis exceeds by the estimation precision the physical basis more than twice.

5. CONCLUSION

The choice of coordinate basis in the signal reconstruction problems has the features which are caused by inaccessibility of signals for direct observations. Optimum properties by criterion of minimum statistical error of measurements are pertained to the PIC- basis.

However, immediate practical usage of the PIC-basis is not always convenient. In this connection the combined technique for forming a coordinate basis is proposed which unites the advantages of physical approach (obviousness, the little dimension of a basis) with advantages of the statistical-information approach (minimization of statistical errors). The given technique is grounded on projection of an arbitrary coordinate basis on the PIC-subspace. As a result the range of possible fluctuations of a signal estimate may be restricted and upper bound of a statistical error is decreased.

The effectiveness of proposed methods is illustrated for the problem of the SSP reconstruction in ocean medium.

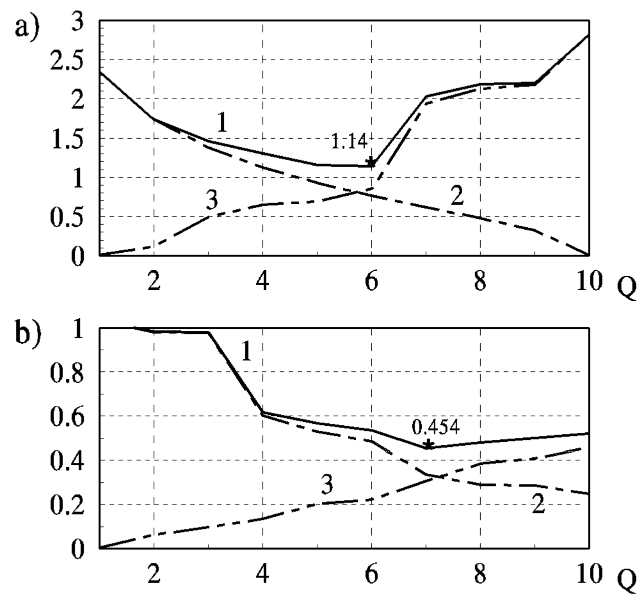

Figure 2: Dependence of the SSP estimation errors on the dimension of the physical (a) and projective (b) basis:
1 - total error, 2 - systematic error, 3 - statistical error.

6. REFERENCES

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