SAMPLING CRITERION FOR NONLINEAR SYSTEMS WITH A BANDPASS INPUT

Ching-Hsiang Tseng

Department of Electrical Engineering
National Taiwan Ocean University
2 Pei-ning Rd., Keelung, Taiwan, R.O.C.

ABSTRACT

Sampling requirements for nonlinear systems with a bandpass input are developed in this paper. It is well known that the output of a nonlinear system may have a larger bandwidth than that of the input. According to the Nyquist sampling theorem, the sampling rate needs to be at least twice the maximum frequency of the output to avoid aliasing. However, if the input is a bandpass signal, the spectrum of the output is usually distributed over several frequency bands. In this case, using the bandpass sampling concept, it is possible to sample the output at a much lower rate. In this paper, conditions for such a lower sampling rate to exist are derived for nonlinear systems up to the third order. Supporting computer simulation is also provided.

1. INTRODUCTION

Nonlinear systems often have a spectral spreading effect on their input signals. For an N-th order nonlinear system, the maximum frequency of the output signal can be as large as N times the maximum frequency of the input signal. In general, the output signal of the nonlinear system needs to be sampled at a rate which is at least $2N$ times the maximum frequency of the system input, otherwise the output samples would be aliased [1]. It has been shown that, for identification of nonlinear systems, it suffices for the sampling rate of the output signal to be twice the maximum frequency of the input signal, even though the resulting output samples are aliased [2, 3]. However, for situations where retaining all information in the output signal is the main concern and system identification is of little interest, avoiding aliasing in the output samples to ensure proper reconstruction of the original output signal is desired.

In many cases, such as digital communications over nonlinear channels [4], etc., the input to the nonlinear system is a bandpass signal whose spectrum is as that shown in Fig. 1(a). The bandpass signal has no spectral components below f_L Hz or above f_H Hz, and has a bandwidth of $B = f_H - f_L$. The bandpass sampling theorem [5] states that such a bandpass signal can be sampled at a frequency f_s without causing aliasing if f_s satisfies

$$\frac{2QB}{n} \leq f_s \leq \frac{2(Q-1)B}{n-1}, \tag{1}$$

where $Q = f_H/B$, and n is a positive integer and $n \leq Q$. For $n = 1$, eq. (1) simply states the Nyquist sampling criterion. For $n > 1$, f_s is smaller than the Nyquist sampling frequency. The larger the value of n, the smaller the sampling frequency. Note that the output of the nonlinear system to such a bandpass signal may possess multiple frequency bands. We see that (1) can not be applied to the output signal.

It is the objective of this paper to derive the sampling requirements for the output signal of nonlinear systems excited by a bandpass input signal. Considering that the cubically nonlinear system is the lowest-order nonlinear system including both even and odd order nonlinear terms, we derive the bandpass sampling requirements for nonlinear systems up to the third order. In fact, one may find in the literature that many nonlinear effects in science and engineering can be appropriately modeled as a cubically nonlinear system. Extension of the derivation to higher-order nonlinear systems, although may be quite involved, is conceptually straightforward.

2. BANDPASS SAMPLING FOR QUADRATICALLY NONLINEAR SYSTEMS

Consider a quadratically nonlinear system (including linear and quadratic responses) with an input $x(t)$ and an output $y(t)$. Suppose that $x(t)$ is a bandpass signal whose spectrum, say $X(f)$, is shown in Fig. 1(a), then the spectrum of the the output $y(t)$, say $Y(f)$, would have a form like Fig. 1(b). Note that, in Fig. 1(b), the spectrum labeled '1' is owing to the linear response of the system, and the spectra labeled '0' and '2' are owing to the quadratic response of the system.

This work was supported in part by the National Science Council under Contracts NSC 89-2611-E-019-048 and NSC 89-2218-E-019-011.

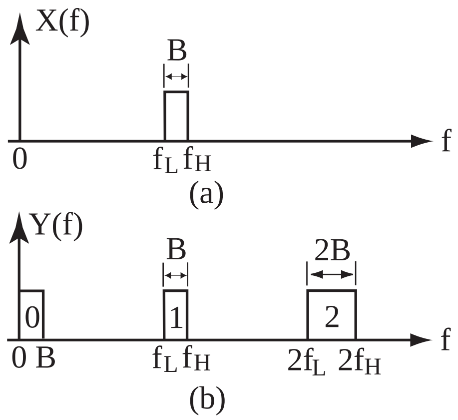

Fig. 1. The spectra of (a) the input and (b) the output signals of a quadratically nonlinear system.

If the output is sampled at f_s Hz, then the spectrum of the sampled output can be obtained by replicating the spectrum of the original signal at multiples of f_s. To be immune from aliasing, neither spectrum '1' nor spectrum '2' can straddle nf_s or $(n+1/2)f_s$ for any integer n. Because if a spectrum (say spectrum '1') straddles nf_s, then its counterpart in the negative frequency region (labeled as spectrum '-1') must straddle $-nf_s$. Due to spectral replication, the spectrum of the sampled output would have the positive and the negative frequency spectra overlapping in the vicinity of every multiple of f_s. This leads to aliasing in the sampled output. Similar result can be reached for the case of a spectrum straddling $(n+1/2)f_s$.

Let the center frequency of spectrum 'i' be f_i, $i = \pm 1$, ± 2. Note that $f_2 = 2f_1$. We must have $nf_s < f_1 < (n+1)f_s$ (and thus $2nf_s < f_2 < 2(n+1)f_s$) for some n. Referring to the frequency band $[kf_s, (k+1)f_s]$ (k an integer) as $segment$ k, we then define ϵ_i as the relative position of f_i in its corresponding segment. For example, f_1 is in segment n, therefore $\epsilon_1 = f_1 - nf_s$. The center frequency f_2 can be either in segment $2n$ or segment $2n+1$, therefore, $\epsilon_2 = R_{f_s}\{f_2 - 2nf_s\} = R_{f_s}\{2\epsilon_1\}$, where $R_a\{b\}$ denotes the remainder of b divided by a.

Due to spectral replication, a replica of spectrum 'i' ($i = \pm 1, \pm 2$) will appear in each segment. It is easy to see that the positions of the replicas '1' and '-1' will be symmetric with respect to the center frequency $f_s/2$ in each segment, so will the positions of the replicas '2' and '-2'. Note that ϵ_i is in fact the relative position of the center frequency of the replica 'i' in each segment. To be immune from aliasing, the replicas in each segment can not overlap. To derive the sampling requirements, it is sufficient to consider the range $0 < \epsilon_1 < f_s$, which is further divided into the following 6 subranges:

Case 1: $0 < \epsilon_1 < \frac{1}{4}f_s$, and hence $0 < \epsilon_2 < \frac{1}{2}f_s$

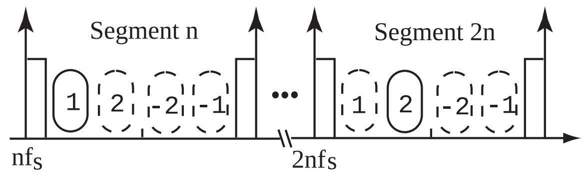

Fig. 2. The spectrum of the sampled output for case 1.

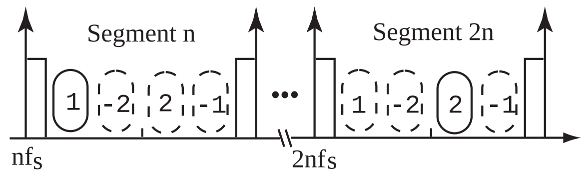

Fig. 3. The spectrum of the sampled output for case 2.

Recall that '1' and '-1' (as well as '2' and '-2') are symmetric with respect to $f_s/2$ in each segment, we have $\frac{3}{4}f_s < \epsilon_{-1} < f_s$ and $\frac{1}{2}f_s < \epsilon_{-2} < f_s$. Since $\epsilon_1 < 2\epsilon_1 = \epsilon_2$, the spectrum of the sampled output would have a form like Fig. 2. In Fig. 2, we use solid circles to denote the spectra of the original output signal, and use dashed circles to denote the spectra of the replicas. The number in each circle indicates where in Fig. 1(b) it is from. For example, a dashed circle labeled '-2' denotes a replica of the spectrum '-2'.

From Fig. 2 we see that the requirements for no spectral overlapping are

$$f_L \geq nf_s + B \quad (2)$$
$$2f_L \geq f_H + nf_s \quad (3)$$
$$2f_H \leq (2n + \frac{1}{2})f_s \quad (4)$$

which leads to

$$\frac{4QB}{4n+1} \leq f_s \leq \frac{(Q-2)B}{n} \quad (5)$$

Case 2: $\frac{1}{4}f_s < \epsilon_1 < \frac{1}{3}f_s$, and hence $\frac{1}{2}f_s < \epsilon_2 < \frac{2}{3}f_s$

In this case, we have $\frac{2}{3}f_s < \epsilon_{-1} < \frac{3}{4}f_s$ and $\frac{1}{3}f_s < \epsilon_{-2} < \frac{1}{2}f_s$. The resulting spectrum of the sampled output is shown in Fig. 3. We see from Fig. 3 that the requirements for no spectral overlapping are

$$f_L \geq nf_s + B \quad (6)$$
$$2f_H \leq (2n+1)f_s - [(f_H - nf_s)] \quad (7)$$
$$2f_L \geq (2n + \frac{1}{2})f_s \quad (8)$$

which yields

$$\frac{3QB}{3n+1} \leq f_s \leq \frac{4(Q-1)B}{4n+1} \quad (9)$$

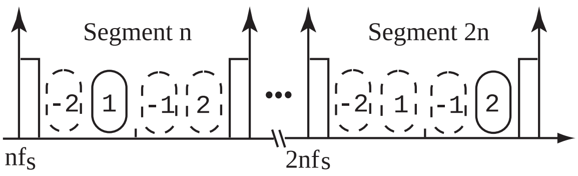

Fig. 4. The spectrum of the sampled output for case 3.

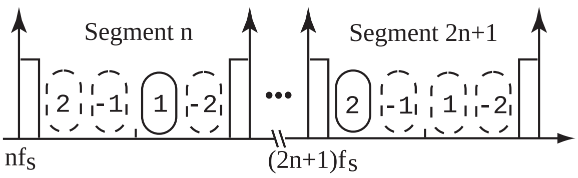

Fig. 5. The spectrum of the sampled output for case 4.

Case 3: $\frac{1}{3}f_s < \epsilon_1 < \frac{1}{2}f_s$, and hence $\frac{2}{3}f_s < \epsilon_2 < f_s$.

In this case, we have $\frac{1}{2}f_s < \epsilon_{-1} < \frac{2}{3}f_s$ and $0 < \epsilon_{-2} < \frac{1}{3}f_s$. Therefore, the spectrum of the resulting sampled output should be like that shown in Fig. 4. The requirements for preventing spectral overlapping in this case are as follows:

$$\begin{aligned} 2f_H &\leq (2n+1)f_s - B & (10)\\ f_L &\geq nf_s + [(2n+1)f_s - 2f_L] & (11)\\ f_H &\leq (n+\frac{1}{2})f_s & (12) \end{aligned}$$

This results in

$$\frac{(2Q+1)B}{2n+1} \leq f_s \leq \frac{3(Q-1)B}{3n+1} \quad (13)$$

Case 4: $\frac{1}{2}f_s < \epsilon_1 < \frac{2}{3}f_s$, and hence $0 < \epsilon_2 < \frac{1}{3}f_s$.

In this case, we have $\frac{1}{3}f_s < \epsilon_{-1} < \frac{1}{2}f_s$ and $\frac{2}{3}f_s < \epsilon_{-2} < f_s$. Note that since $f_s < 2\epsilon_1 < \frac{4}{3}f_s$, the spectrum '2' is in segment $2n+1$ instead of $2n$. Therefore, the spectrum of the sampled output for this case is shown in Fig. 5. To avoid spectral overlapping, we must have

$$\begin{aligned} 2f_L &\geq (2n+1)f_s + B & (14)\\ f_H &\leq (n+1)f_s - [2f_H - (2n+1)f_s] & (15)\\ f_L &\geq (n+\frac{1}{2})f_s & (16) \end{aligned}$$

which gives the following

$$\frac{3QB}{3n+2} \leq f_s \leq \frac{(2Q-3)B}{2n+1} \quad (17)$$

Case 5: $\frac{2}{3}f_s < \epsilon_1 < \frac{3}{4}f_s$, and hence $\frac{1}{3}f_s < \epsilon_2 < \frac{1}{2}f_s$.

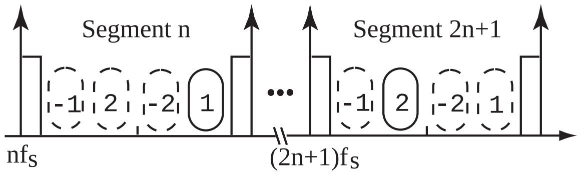

Fig. 6. The spectrum of the sampled output for case 5.

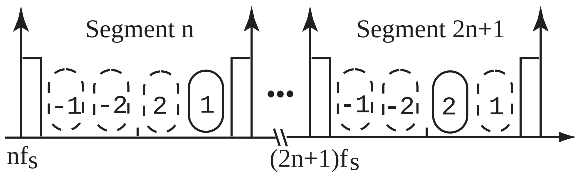

Fig. 7. The spectrum of the sampled output for case 6.

In this case, we have $\frac{1}{4}f_s < \epsilon_{-1} < \frac{1}{3}f_s$ and $\frac{1}{2}f_s < \epsilon_{-2} < \frac{2}{3}f_s$. Since $\frac{4}{3}f_s < 2\epsilon_1 < \frac{3}{2}f_s$, the spectrum '2' is in segment $2n+1$. The spectrum of the sampled output for this case is shown in Fig. 6. The requirements for no spectral overlapping in Fig. 6 are

$$\begin{aligned} f_H &\leq (n+1)f_s - B & (18)\\ 2f_L &\geq (2n+1)f_s + [(n+1)f_s - f_L] & (19)\\ 2f_H &\leq (2n+\frac{3}{2})f_s & (20) \end{aligned}$$

which leads to

$$\frac{4QB}{4n+3} \leq f_s \leq \frac{3(Q-1)B}{3n+2} \quad (21)$$

Case 6: $\frac{3}{4}f_s < \epsilon_1 < f_s$, and hence $\frac{1}{2}f_s < \epsilon_2 < f_s$

In this case, we have $0 < \epsilon_{-1} < \frac{1}{4}f_s$ and $0 < \epsilon_{-2} < \frac{1}{2}f_s$. Since $\frac{3}{2}f_s < 2\epsilon_1 < 2f_s$, the spectrum '2' is again in segment $2n+1$. It is easy to see that, in this case, $\epsilon_2 = R_{f_s}\{2\epsilon_1\}$ must be smaller than ϵ_1, hence the spectrum of the sampled output is as that shown in Fig. 7.

To avoid spectral overlapping in Fig. 7, we must have

$$\begin{aligned} f_H &\leq (n+1)f_s - B & (22)\\ 2f_H &\leq f_L + (n+1)f_s & (23)\\ 2f_L &\geq (2n+\frac{3}{2})f_s & (24) \end{aligned}$$

which results in

$$\frac{(Q+1)B}{n+1} \leq f_s \leq \frac{4(Q-1)B}{4n+3} \quad (25)$$

3. BANDPASS SAMPLING FOR CUBICALLY NONLINEAR SYSTEMS

Consider a cubically nonlinear system (which includes linear, quadratic, and cubic responses). Suppose the input to

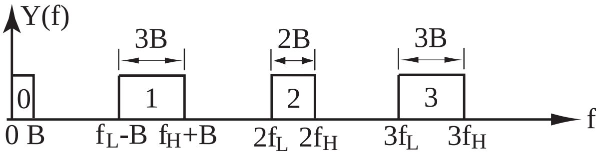

Fig. 8. The spectrum of the output signal of a cubically nonlinear system.

the system is a bandpass signal whose spectrum is shown in Fig. 1(a), then the output of the system would have a spectrum like the one shown in Fig. 8. Note that the spectra labeled '0' and '2' are owing to the quadratic response, the spectrum labeled '3' is owing to the cubic response, and the spectrum labeled '1' is owing to the linear as well as the cubic responses. Following similar steps as shown in Section 2 and with some algebra, the bandpass sampling requirements can be derived. The result is summarized in the following table, where valid sampling frequency ranges for various possible orders of spectra in each segment are shown.

Spectral Order	Sampling Frequency Range
$\{1,2,3,-3,-2,-1\}$	$\frac{6QB}{6n+1} \leq f_s \leq \frac{(Q-3)B}{n}$
$\{1,2,-3,3,-2,-1\}$	$\frac{5QB}{5n+1} \leq f_s \leq \frac{6(Q-1)B}{6n+1}$
$\{1,-3,2,-2,3,-1\}$	$\frac{(4Q+1)B}{4n+1} \leq f_s \leq \frac{5(Q-1)B}{5n+1}$
$\{-3,1,-2,2,-1,3\}$	$\frac{(3Q+1)B}{3n+1} \leq f_s \leq \frac{(4Q-5)B}{4n+1}$
$\{3,-2,1,-1,2,-3\}$	$\frac{5QB}{5n+2} \leq f_s \leq \frac{(3Q-4)B}{3n+1}$
$\{-2,3,1,-1,-3,2\}$	$\frac{2(Q+1)B}{2n+1} \leq f_s \leq \frac{5(Q-1)B}{5n+2}$
$\{2,-3,-1,1,3,-2\}$	$\frac{5QB}{5n+3} \leq f_s \leq \frac{2(Q-2)B}{2n+1}$
$\{-3,2,-1,1,-2,3\}$	$\frac{(3Q+1)B}{3n+2} \leq f_s \leq \frac{5(Q-1)B}{5n+3}$
$\{3,-1,2,-2,1,-3\}$	$\frac{(4Q+1)B}{4n+3} \leq f_s \leq \frac{(3Q-4)B}{3n+2}$
$\{-1,3,-2,2,-3,1\}$	$\frac{5QB}{5n+4} \leq f_s \leq \frac{(4Q-5)B}{4n+3}$
$\{-1,-2,3,-3,2,1\}$	$\frac{6QB}{6n+5} \leq f_s \leq \frac{5(Q-1)B}{5n+4}$
$\{-1,-2,-3,3,2,1\}$	$\frac{(Q+2)B}{n+1} \leq f_s \leq \frac{6(Q-1)B}{6n+5}$

4. SIMULATION

A bandpass signal ($f_L = 990$ Hz, $f_H = 1000$ Hz, and $B = 10$ Hz) is used as the input to the cubically nonlinear system $y(t) = x(t) + x^2(t) + x^3(t)$. The spectrum of the system output is shown in Fig. 9(a), where 4 frequency bands at around 0, 1000, 2000, and 3000 Hz are clearly shown. Using $n = 3$, we found $273.64 < f_s < 275$ is the valid range for the case of spectral order $\{-3, 2, -1, 1, -2, 3\}$. We chose $f_s = 274$ Hz to sample the output signal. The spectrum of the sampled output is shown in Fig. 9(b). The result indicates that various spectra do not overlap, and the spectral order is indeed $\{-3, 2, -1, 1, -2, 3\}$.

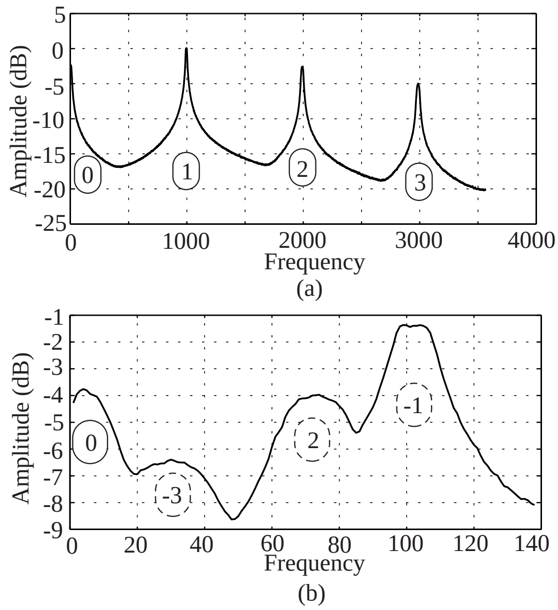

Fig. 9. Spectra of (a) the original and (b) the sampled output signals in the simulation.

5. CONCLUSION

Bandpass sampling requirements for nonlinear systems up to the third order were derived. The result can be used to select appropriate sampling frequency for the output of nonlinear systems with a bandpass input.

6. REFERENCES

[1] S. W. Nam and E. J. Powers, "Application of higher order spectral analysis to cubically nonlinear system identification," *IEEE Trans. Signal Processing*, vol. 42, no. 7, pp. 1746-1765, July 1994.

[2] R. J. Martin, "Volterra System Identification and Kramer's Sampling Theorem," *IEEE Trans. Signal Processing*, vol. 47, no. 11, pp. 3152-3155, Nov. 1999.

[3] C.-H. Tseng, "Identification of cubically nonlinear systems using undersampled data," *IEE Proc.-Vis. Image Signal Process.*, vol. 144, no. 5, pp. 267-277, Oct. 1997.

[4] K. W. Schneider and W. H. Tranter, "Efficient Simulation of Multicarrier Digital Communication Systems in Nonlinear Channel Environments," *IEEE J. Select. Areas Commun.*, vol. 11, no. 3, pp. 328-339, April 1993.

[5] R. G. Vaughan, N. L. Scott, and D. R. White, "The theory of bandpass sampling," *IEEE Trans. Signal Processing*, vol. 39, no. 9, pp. 1973-1984, Sept. 1991.