MAXIMUM-SNR SPACE-TIME DESIGNS FOR MIMO CHANNELS

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ABSTRACT

We consider a communication scenario involving an $m \times n$ MIMO linear channel whose input is a symbol stream multiplied prior to transmission by an $n \times \bar{n}$ space-time coding matrix X, and whose output is fed into an $m \times \bar{n}$ linear combiner Z. We show how to choose the matrices X and Z to maximize the SNR of the linear combiner output data that are used for detection, under total power constraint (TPC), elemental power constraint (EPC), or total and elemental power constraint (TEPC). The TEPC design (considered here for the first time) is shown to include the TPC and EPC designs (previously considered by the authors) as special cases, and hence to provide a theoretically and practically interesting unifying framework. We make use of this framework to discuss various tradeoffs of the three space-time designs considered, such as transmission rate and requirements for channel status information at the transmission side.

1. INTRODUCTION AND PROBLEM FORMULATION

Let s be the (scalar) symbol to be transmitted, X the $n \times \bar{n}$ space-time coding matrix, and A the $m \times n$ transfer matrix of the communication channel. Then the channel output can be written as

$$Y = AXs + N \tag{1}$$

where N is an $m \times \bar{n}$ noise term. The above equation is valid for flat fading channels (see, e.g, [2]). However for (1) to hold the channel is required to be only slowly time-varying so that A is (nearly) constant during the transmission of Xs.

Under the Gaussian hypothesis the maximum likelihood detector of the symbol stream is a linear function of the received data. Let Z^* be an $\bar{n} \times m$ linear combiner matrix used at the receiver to form a general linear combination of the elements of Y, namely $tr(Z^*Y)$. Hereafter $*$ denotes the conjugate transpose and $tr(\cdot)$ denotes the trace operator. The detection of s is based on $tr(Z^*Y)$ which in view of (1) satisfies the equation:

$$tr(Z^*Y) = tr(Z^*AX)s + tr(Z^*N) \tag{2}$$

Under the assumption that the elements of N are i.i.d random variables with mean zero and common variance σ^2, that is,

$$E(N_{ij}N^*_{kp}) = \sigma^2 \delta_{ik}\delta_{jp} \tag{3}$$

one can readily check that

$$E|tr(Z^*N)|^2 = \sigma^2 tr(Z^*Z) \tag{4}$$

Hence the SNR for the detection data in (2) is

$$\text{SNR} = \frac{|tr(Z^*AX)|^2}{\sigma^2 tr(Z^*Z)}\sigma_s^2 \tag{5}$$

where $\sigma_s^2 = E|s|^2$. By the Cauchy-Schwartz matrix inequality (see, e.g, [1]):

$$\text{SNR} \leq \frac{\sigma_s^2}{\sigma^2} tr(X^*A^*AX) \tag{6}$$

where the equality is achieved for:

$$Z = AX \tag{7}$$

Note that, as expected, the maximum SNR linear combiner is a *matched filter*. Let:

$$R = XX^* \tag{8}$$

It follows from (6) that the problem left is to choose X so that

$$\overline{\text{SNR}} = \frac{\sigma_s^2}{\sigma^2} tr(RA^*A) \tag{9}$$

is maximized under a suitable power constraint (to prevent the impractical solution $\|R\| \to \infty$). Solving this problem under various power constraints is the main topic dealt with in the remainder of the paper.

2. POWER CONSTRAINTS

In what follows $\lambda_{max}(B)$ denotes the maximum eigenvalue of the matrix B.

2.1. Total Power Constraint (TPC)

$$tr(R) \leq \alpha > 0 \tag{10}$$

This is a commonly-used constraint and hence requires no comment.

This work was partly supported by the Senior Individual Grant Program of the Swedish Foundation for Strategic Research (SSF).

2.2. Elemental Power Constraint (EPC)

$$\lambda_{max}(R) \leq \beta > 0 \tag{11}$$

This is a less-often used power constraint and hence some comments on it are in order. Since $\{R_{ii}\}$, the diagonal elements of R, are upper-bounded by $\lambda_{max}(R)$,

$$R_{ii} \leq \lambda_{max}(R) \tag{12}$$

it follows that indeed (11) is a constraint on the elemental powers $\{R_{ii}\}$. However it is only an indirect constraint in the sense that the matrix R may well be such that $R_{ii} < \beta$ (strictly) even when $\lambda_{max}(R) = \beta$. Despite this fact we prefer (11) to the more direct elemental power constraints $\{R_{ii} \leq \beta\}$ because the EPC design derived from (11) has a simple form (see below).

2.3. Total and Elemental Power Constraint (TEPC)

$$\begin{aligned} tr(R) &\leq \alpha \\ \lambda_{max}(R) &\leq \beta \end{aligned} \tag{13}$$

As

$$\lambda_{max}(R) \leq tr(R) \leq n\lambda_{max}(R) \tag{14}$$

it follows that:

1. If $\beta \geq \alpha$ then only the TPC is active (that is, the EPC is implied by the TPC for this choice of α and β), and hence the TEPC reduces to TPC;
2. If $n\beta \leq \alpha$ then only the EPC is active and TEPC reduces to EPC; and
3. If $\frac{1}{n}\alpha < \beta < \alpha$ both constraints in (13) are active in general.

In the sequel we derive the maximum SNR designs for R (or, essentially equivalent, for X) under the three constraints above. The TPC and EPC designs are presented without a proof since they were previously derived in [2, 3] and, moreover, they follow as special cases of the *novel* TEPC design (as explained in 1 and 2 above; also see the next section).

3. MAXIMUM SNR DESIGNS

Let $r = \text{rank}(A)$ and let

$$A^*A = U\Lambda U^* \tag{15}$$

denote the eigenvalue decomposition (EVD) of A^*A, where U is an $n \times r$ semi-unitary matrix ($U^*U = I$)

$$U = [u_1 \ \dots \ u_r] \tag{16}$$

and

$$\Lambda = \begin{bmatrix} \lambda_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & \lambda_r \end{bmatrix} \tag{17}$$

with $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_r > 0$.

3.1. TPC Design

The matrix R that maximizes the $\overline{\text{SNR}}$ in (9) under the TPC is (see [2] and Section 3.4):

$$R_{TPC} = \alpha u_1 u_1^* \tag{18}$$

The corresponding maximum SNR is given by

$$\text{SNR}_{TPC} = \frac{\sigma_s^2}{\sigma^2}\alpha\lambda_1 \tag{19}$$

If we consider X to be an $n \times \bar{n}$ matrix then it is obvious that the TPC design exists for any $\bar{n} \geq 1$. In particular we can use it with $\bar{n} = 1$ to achieve full transmission rate, in which case:

$$X_{TPC} = (\alpha)^{1/2}u_1 \tag{20}$$

This is not possible for the EPC design discussed next.

3.2. EPC Design

The maximum SNR under the EPC is

$$\text{SNR}_{EPC} = \frac{\sigma_s^2}{\sigma^2}\beta tr(A^*A) \tag{21}$$

and it is achieved at (see [2] and Section 3.4):

$$R_{EPC} = \beta I \tag{22}$$

where I is the $n \times n$ identity matrix.

The great advantage of the EPC design, which is not shared by any other design discussed in this paper, is that *channel information is not required at the transmission side.* However, unlike the TPC design and the TEPC design (discussed in Section 3.4), the EPC design in (22) requires at least $\bar{n} = n$ to exist, in which case

$$X_{EPC} = (\beta)^{1/2}I \tag{23}$$

This is a drawback since it leads to a reduction in the transmission rate of the EPC design by a factor of n as compared with the TPC design. *Space-Time Block Codes* (STBC) may be used to increase the transmit rate of the EPC design (see [2, 3, 6] for details). However recovering full data rates by using the currently available STBC is only possible for very few values of n (such as $n = 2$), for most other values we can only achieve 1/2 of the maximum possible data rate (see the cited references).

3.3. Maxmin Interpretation of the EPC Design

As explained above, an important feature of the EPC design is that it does not need channel information at the transmission side, even though that was *not* an *a priori* design requirement. In the following we formulate a *maxmin* optimization problem in which we first derive the *worst channel* that minimizes the $\overline{\text{SNR}}$ in (9) over a general class of channel matrices A and then obtain the matrix R that maximizes the worst-case $\overline{\text{SNR}}$ *under the TPC*. Mathematically,

$$\max_R \min_A tr(RA^*A) \tag{24}$$

subject to:

$$A \in C = \{A | \lambda_r \geq \rho\} \quad (25)$$
$$tr(R) \leq \alpha \quad (26)$$

In (25), r is the rank of A, λ_r is the smallest non-zero eigenvalue of A^*A (see (15)-(17)), and $\rho > 0$ is an *arbitrary* constant. By choosing ρ sufficiently small we can evidently include any given possible A in the above class of channels C. Note that the constraint $A \in C$ is needed to eliminate the worst trivial channel $A = 0$. Also note that the maxmin optimal R matrix will not depend on A, by design. It is shown in [5] that the *maxmin optimal matrix R is*:

$$R_{maxmin} = (\alpha/n)I \quad (27)$$

which coincides with the EPC design in (22) with $\beta = \alpha/n$ (to satisfy the TPC considered here, see (26)). The maxmin interpretation is an interesting property of the EPC design. In words it says that for any given design that is *channel independent* and satisfies the TPC we can find a matrix A so that the corresponding $\overline{\text{SNR}}$ is smaller than that associated with (27).

3.4. TEPC Design

This design is the solution to the following optimization problem:

$$\max_R tr(RA^*A) \quad (28)$$

subject to:

$$tr(R) \leq \alpha$$
$$\lambda_{max}(R) \leq \beta \quad (29)$$

To solve the above problem we make use of the EVD of A^*A in (15)-(17) and of the EVD of R

$$R = V\Gamma V^* \quad (30)$$

where V is an $n \times n$ unitary matrix and:

$$\Gamma = \begin{bmatrix} \gamma_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & \gamma_n \end{bmatrix} \quad (31)$$

The eigenvalues $\{\gamma_k\}$ of R are ordered as stated in the previous subsection: $\gamma_1 \geq \ldots \geq \gamma_n \geq 0$. By a corollary of von Neumann theorem in [4] we have that:

$$tr(RA^*A) \leq \sum_{k=1}^{r} \lambda_k \gamma_k \quad (32)$$

The upper-bound in (32) is obviously achieved for:

$$R_{TEPC} = U \begin{bmatrix} \gamma_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & \gamma_r \end{bmatrix} U^* \quad (33)$$

It remains to obtain the $\{\gamma_k\}$ that solve the problem:

$$\max_{\{\gamma_k\}} \sum_{k=1}^{r} \lambda_k \gamma_k \quad (34)$$

subject to:

$$\sum_{k=1}^{r} \gamma_k \leq \alpha \quad (35)$$
$$\gamma_1 \leq \beta \quad (36)$$

Let $\bar{r}$ denote the integer part of α/β:

$$\bar{r} = \lfloor \alpha/\beta \rfloor \quad (37)$$

Under the constraints in (35)(36),

$$\sum_{k=1}^{r} \lambda_k \gamma_k \leq \begin{cases} \beta \sum_{k=1}^{\bar{r}} \lambda_k + (\alpha - \bar{r}\beta)\lambda_{\bar{r}+1} & \text{for } \bar{r} < r \\ \beta \sum_{k=1}^{r} \lambda_k & \text{for } \bar{r} \geq r \end{cases} \quad (38)$$

The upper-bound in (38) is achieved for:

$$\left.\begin{array}{l} \gamma_1 = \ldots = \gamma_{\bar{r}} = \beta,\ \gamma_{\bar{r}+1} = \alpha - \bar{r}\beta, \\ \gamma_{\bar{r}+2} = \ldots = \gamma_r = 0 \end{array}\right\} \quad \text{for } \bar{r} < r$$
$$\gamma_1 = \ldots = \gamma_r = \beta \qquad \text{for } \bar{r} \geq r \quad (39)$$

The corresponding maximum SNR is:

$$\text{SNR}_{TEPC} = \frac{\sigma_s^2}{\sigma^2} \sum_{k=1}^{r} \lambda_k \gamma_k \quad (40)$$

In summary, then, the *TEPC design* is given by (33),(37) and (39). Note that for $\bar{r} > r$ we do not use the total power allowed, α, but only a fraction of it equal to βr.

Like the TPC design, the TEPC design requires channel information at the transmission side. However, unlike the TPC design, the TEPC design does not achieve full transmission rate: indeed, a direct implementation of (33) would reduce the transmission rate by a factor equal to $min(r, \bar{r}+1)$; note that the X matrix, with minimum number of columns, associated with (33) is:

$$X_{TEPC} = U\Gamma^{1/2} \quad (41)$$

where

$$\Gamma^{1/2} = \begin{bmatrix} \gamma_1^{1/2} & \dots & 0 \\ & \ddots & \\ 0 & \dots & \gamma_r^{1/2} \end{bmatrix}$$

But since we know the channel at the transmitter we can modify the scheme so as to transmit simultaneously r symbols to achieve the full data rate and still achieve the maximum SNR corresponding to the TEPC design for each of the transmitted symbols [5].

We remind the reader that the EPC design can also be modified to achieve full rate but only for $n = 2$. For all other values of n the maximum rate that can be achieved by the EPC design via the use of the currently available STBC is less than full (3/4, or most often 1/2) [3, 2].

Finally we note that (as explained in Section 2.3) it is possible to obtain the TPC design as a special case of the TEPC design simply by choosing $\beta > \alpha$. This is in fact easily checked, as for $\beta > \alpha$ we have $\bar{r} = 0$ and hence, from (33) and (39)

$$R = \alpha u_1 u_1^*$$

which coincides with the TPC design (18).

Similarly, the EPC design results as a special case of the TEPC design if $\alpha \geq n\beta$ (see Section 2.3). To see this, first note that $\alpha \geq n\beta$ implies $\bar{r} \geq n \geq r$. Hence the TEPC design becomes:

$$R = \beta U U^* \tag{42}$$

which depends on the channel. However the same SNR value (proportional to $\sum_{k=1}^{r} \lambda_k = tr(A^*A)$) can evidently be achieved by using the channel-independent EPC design:

$$R = \beta I \tag{43}$$

(cf. (22)). It is interesting to observe the tradeoffs made when using (43) in lieu of the alternate design (42): we eliminate the need for channel information at the transmission side, but we lose transmission rate (as explained before) and also we spend more transmit power (the total power for (43) is $n\beta$ whereas it is $r\beta$ for (42)).

4. NUMERICAL EXAMPLES

In this section we present two numerical examples for the TEPC design (numerical examples for the EPC and TPC designs can be found in [2]). The channel we consider is a flat Rayleigh fading channel. The elements of A, $\{A_{ij}\}$, are considered to be i.i.d complex Gaussian random variables with mean zero and variance equal to one: $A_{ij} \sim \mathcal{N}(0, 1)$. The variance of the additive Gaussian noise as well as the signal power are set to one: $\sigma^2 = \sigma_s^2 = 1$. We want to illustrate numerically the average SNR associated with the TEPC design.

Example **1** *In this example we consider the TEPC design for a fixed α and varying β. We set the number of transmitter antennas equal to the number of receiver antennas ($m = n$). A is assumed to be full rank, i.e. $r = n$. Also we set $\alpha = 1$ and vary β between 0.1 and 1.5 in steps of 0.1. The average (over A) SNR_{TEPC} is plotted versus β in Figure 1 for different values of $m = n$. The averages were computed over 1000 realizations of A. From Figure 1 it can be seen that when $\beta = \alpha$ the EPC constraint becomes inactive and there is no increase in SNR as β is increased further.*

Example **2** *In this example we consider the TEPC design for a fixed β and varying α. Again we set $m = n$ and assume that A has full rank $r = n$. We set $\beta = 0.5$ and vary α between 0.1 and 2.5 in steps of 0.1. The average (over A) SNR_{TEPC} is plotted versus α in Figure 2 for different values of $m = n$. As for Example 1, the averages were computed over 1000 realizations of A. It can be seen from Figure 2 that when $\alpha = n\beta$ the TPC constraint becomes inactive and there is no increase in SNR as α is increased further.*

5. REFERENCES

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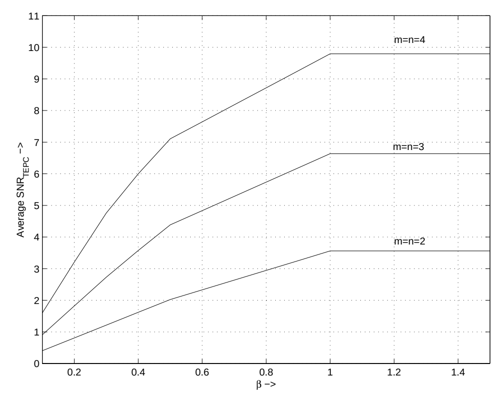

Figure 1: The (average) SNR_{TEPC} versus β for $\alpha = 1$.

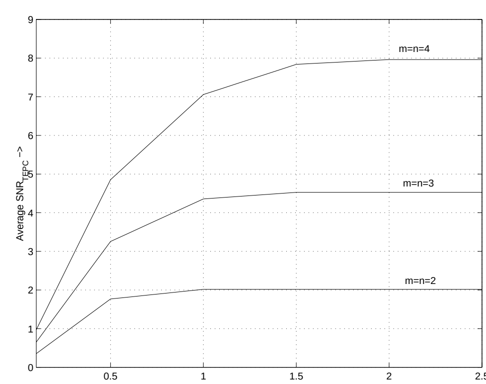

Figure 2: The (average) SNR_{TEPC} versus α for $\beta = 0.5$.