A FAMILY OF COSINE-SUM WINDOWS FOR HIGH-RESOLUTION MEASUREMENTS

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ABSTRACT

Special windows are used in spectral analysis to reduce the effect of spectral leakage. Windows with low sidelobe amplitude are necessary for the detection of small signals when highly dynamic spectra are concerned. The design of a family of cosine-sum windows with minimum sidelobes is described. The coefficients and selected parameters for windows with a peak sidelobe level between 43 dB and 289 dB are stated.

1. INTRODUCTION

Window functions are used when a section of length T_w is to be cut from a time varying signal for the purpose of spectral analysis. If the signal is not periodic, or if T_w is not an integer multiple of the signal period, spectral leakage occurs. This may result in obscuring neighbouring spectral terms of small amplitude.

This is a major problem in the investigation of high-resolution analog to digital converters (ADCs) using sinusoidal test signals. Because the requirements for spectral purity of the test signal are extreme, frequency stability and resolution is traded for low distortion in the signal. If it is therefore impossible to have phase locked coupling between the sine wave test signal and the sample clock, window functions with extremely low sidelobe levels are required for the detection of low amplitude spectral components.

Windows composed of cosine terms reduce the computational complexity in obtaining the weighting coefficients as the window length varies. The windows described in [1–3] show peak sidelobe levels higher than 181 dB. However, for an ideal 24 bit AD converter, the noise level of the Fourier-transformed output signal is lower than 206 dBc (dB carrier) for a data set with 2^{21} samples. The common windows [1, 2] and the 7-term Blackman-Harris window [3] may not meet this requirement.

The design of cosine-sum windows with minimum peak sidelobe level is described in the following. The coefficients and selected parameters are indicated for the 2- to 11-term window.

2. ANALYSIS OF THE COSINE-SUM WINDOW

The weighting factors of the cosine-sum window are as follows:

$$w(t) = \begin{cases} \sum_{p=0}^{G} (\ 1)^p A_p \cos\left(2\pi p \frac{t}{T_w}\right) & \text{for } 0 \le t \le T_w \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

After conversion the following expression is obtained which is better suited for a fast calculation of the weighting factors:

$$w(t) = \begin{cases} \sum_{p=0}^{G} (\ 1)^p B_p \cos^p\left(2\pi \frac{t}{T_w}\right) & \text{for } 0 \le t \le T_w \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The Fourier transform of the window is the spectral window $W(f)$.

$$W(f) = \frac{1}{T_w} \int_0^{T_w} w(t) e^{\ j2\pi f t} dt \tag{3}$$

With $Q = fT_w$, it then follows from (1) and (3)

$$W(Q) = [\sin(2\pi Q) \quad j(1 \quad \cos(2\pi Q))] \frac{1}{2\pi} \sum_{p=0}^{G} (\ 1)^p A_p \frac{Q}{Q^2 \quad p^2} \tag{4}$$

with the limiting values

$$\lim_{Q\to 0} W(Q) = A_0$$
$$\lim_{|Q|\to p} W(Q) = (\ 1)^p \frac{1}{2} A_p \quad \text{for } p = 1, 2, \ldots, G \tag{5}$$

and

$$|W(Q)| = \left| \frac{\sin(\pi Q)}{\pi} \sum_{p=0}^{G} (\ 1)^p A_p \frac{Q}{Q^2 \quad p^2} \right| \tag{6}$$

with the limiting values

$$\lim_{Q\to 0} W(Q) = A_0$$
$$\lim_{|Q|\to p} W(Q) = \frac{1}{2} A_p \quad \text{for } p = 1, 2, \ldots, G \tag{7}$$

With integer Q, zeros of (4) and (6) occur with $|Q| \geq (G+1)$. Additional zeros, whose location is determined by the coefficients A_p, can be produced by the term

$$\sum_{p=0}^{G} (\ \ 1)^p A_p \left(\frac{Q}{Q^2 \ \ p^2} \right) \tag{8}$$

The result in equation (8) can be used to optimize the characteristics of the window.

In the following, we will consider the case of symmetric zeros with respect to $Q = 0$. Equation (8) can be expressed as the quotient of two polynomials. The following results for a real zero pair $\pm Q_k$:

$$(Q + Q_k)(Q \ \ Q_k) = \ Q^2 \ \ Q_k^2) \tag{9}$$

and for $G \geq 1$

$$\sum_{p=0}^{G} (\ \ 1)^p A_p \left(\frac{Q}{Q^2 \ \ p^2} \right) = \frac{V}{Q} \prod_{k=0}^{G\ \ 1} \left(\frac{Q^2 \ \ Q_k^2}{Q^2 \ \ (k+1)^2} \right) \tag{10}$$

Accordingly, equation (4) can be expressed for real zero pairs $\pm Q_k$ in the form

$$W(Q) = \frac{1}{2\pi} \left[\sin(2\pi Q) \ \ j(1 \ \ \cos(2\pi Q)) \right] \frac{V}{Q} \prod_{k=0}^{G\ \ 1} \left(\frac{Q^2 \ \ Q_k^2}{Q^2 \ \ (k+1)^2} \right) \tag{11}$$

with the limiting values

$$\lim_{Q \to 0} W(Q) = V \prod_{k=0}^{G\ \ 1} \left(\frac{Q_k^2}{(k+1)^2} \right)$$
$$\lim_{|Q| \to p} W(Q) = V \frac{\prod_{k=0}^{G\ \ 1} \ p^2 \ \ Q_k^2)}{2 \prod_{\substack{m=0 \\ m \neq p}}^{G} \ p^2 \ \ m^2)} \quad \text{for } p = 1, 2, \ldots, G \tag{12}$$

The amplification of the spectral window is set using the constant V.

3. DESIGN OF MINIMUM SIDELOBE WINDOWS

3.1. Determination of the zeros

For window (1) with $(G+1)$ terms, the lobe of the Fourier transform $W(Q)$ for $|Q| \leq (G+1)$ be the main lobe with the maximum at location $Q = 0$, and the lobes for $|Q| > (G+1)$ be the sidelobes.

The real zero pairs $\pm Q_k$ with $(G+1) < |Q_0| \leq |Q_1| \leq \ldots \leq \left| Q_{(G\ \ 1)} \right|$ are to be chosen so that the maximum of the function

$$|W_{norm}(Q)| = \left| \frac{W(Q)}{W(0)} \right| \tag{13}$$

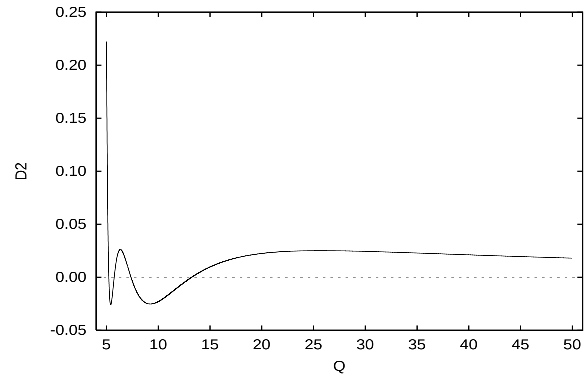

Fig. 1. Plot of $D_2(Q)$ for a 5-term cosine-sum window

becomes minimum for $Q > (G+1)$. Using equations (6), (10) and (12), it then follows:

$$|W_{norm}(Q)| = |D_1 D_2 D_3| \tag{14}$$

with

$$D_1 = \frac{\sin(\pi Q)}{\pi} \qquad D_2 = \frac{1}{Q} \prod_{k=0}^{G\ \ 1} \left(\frac{Q^2 \ \ Q_k^2}{Q^2 \ \ (k+1)^2} \right)$$
$$D_3 = \prod_{k=0}^{G\ \ 1} \left(\frac{(k+1)^2}{Q_k^2} \right) \tag{15}$$

To exploit the symmetry of equation (14), only the case where $Q \geq 0$ with positive zeros will be considered. Term D_2 in equation (15) allows the sidelobes of the window function to be minimized. Between two neighbouring zeros and for $Q > Q_{(G\ \ 1)}$, D_2 shows a local extremum (Fig. 1). When D_1 is multiplied by D_2, at least one additional local extremum is formed for Q with $(G+1) < Q \leq Q_0$. D_3 serves for normalization only. In equation (14), within each of the intervals

$$\left((G+1), Q_0\right], \left[Q_0, Q_1\right], \ldots, \left[Q_{(G\ \ 2)}, Q_{(G\ \ 1)}\right], \left[Q_{(G\ \ 1)}, \infty\right) \tag{16}$$

there is, therefore, a maximum M_k for $k = 0, 1, 2, \ldots, G$. It can be shown that the amplitude of all maxima increases for $(G+1) < Q \leq Q_k$ and that the amplitude of all maxima decreases for $Q > Q_k$ when the Q_k zero is displaced by the value ϵ (and vice versa).

When all maxima have the same amplitude, any displacement of the zero by the value $\pm\epsilon$ results in the amplitude of at least one maximum being increased. It is, therefore, assumed that a minimum of $max(M_0, M_1, \ldots, M_G)$ is reached here.

A simple algorithm is used to optimize the zeros for the same amplitude of all maxima. Suitable starting values are:

$$Q_k = G + k + 2 \quad \text{with } k = 0, 1, 2, \ldots, (G \ \ 1) \tag{17}$$

In every optimization cycle, the maxima M_k and the mean value

$$\overline{M} = \frac{1}{G+1} \sum_{k=0}^{G} M_k \tag{18}$$

are determined. For the maxima M_k with $k < G$, starting with M_0, $Q_k := Q_k - \Delta Q_k$ occurs if $M_k > \overline{M}$, otherwise $Q_k := Q_k + \Delta Q_k$. When $Q_k \leq (G+1)$ or $Q_k >= Q_{(k+1)}$, Q_k retains the original value. This is repeated until $max\,(M_0, M_1, \ldots, M_G)$ no longer becomes significantly smaller. The optimization cycle can be repeated with decreasing ΔQ_K, until $Q_k = (Q_k + \Delta Q_k)$ as a result of too low a resolution of the arithmetic used.

During the first cycles, all ΔQ_k have the same value, and as a result an approximation is quickly found for the zeros Q_k. For further optimization, the ΔQ_k are chosen as a function of the length of the intervals in equation (16).

3.2. Calculation of the window coefficients

When the zeros Q_k are known, for $Q = Q_k$ a system of linear equations for the determination of the coefficients A_p can be set up using equation (10). The residual degree of freedom is used to normalize the coefficients.

Obtaining sufficient numerical accuracy directly from the system of equations presents a difficult problem due to limitations in double precision arithmetic. The coefficients can also be determined from the limiting values of the spectral window. This approach is simple by comparison and it requires less numerical precision. From equations (5) and (12) it follows that

$$A_p = \begin{cases} V \prod_{k=0}^{G-1} \left(\frac{Q_k^2}{(k+1)^2} \right) & \text{for } p = 0 \\ (-1)^p \, V \frac{\prod_{k=0}^{G-1} (p^2 - Q_k^2)}{\prod_{\substack{m=0 \\ m \neq p}}^{G} (p^2 - m^2)} & \text{for } p = 1, 2, \ldots, G \end{cases} \tag{19}$$

The coefficients are normalized using the constant V. For the windows indicated here, V has been chosen such that the condition $\sum_{p=0}^{G} A_p = 1$ is fulfilled.

4. RESULTS

The coefficients have been determined for the 2- to 11-term minimum sidelobe window (Table 1). When the window coefficients are rounded to two decimal places, the 2-term window is identical with the Hamming window [4]. 3- and 4-term windows show good agreement with the coefficients stated in [2].

The algorithm for the optimization of the locations of the zeros and for the calculation of the coefficients of the window functions is implemented using the C programming language [5]. The double type, which corresponds to the double-real format in [6], is available for all calculations as the data type with the highest resolution. It allows the coefficients of window functions with a maximum of 12 terms to be determined. Up to the 11-term window, the amplitude spectra calculated with equation (6) show good agreement with the expected peak side lobe levels. On the basis of the amplitude spectra of the windows obtained by means of FFT, this result can be confirmed (Fig. 2). In the case of the 12-term window, the numerical errors degrade the amplitude spectrum as a result of the finite resolution.

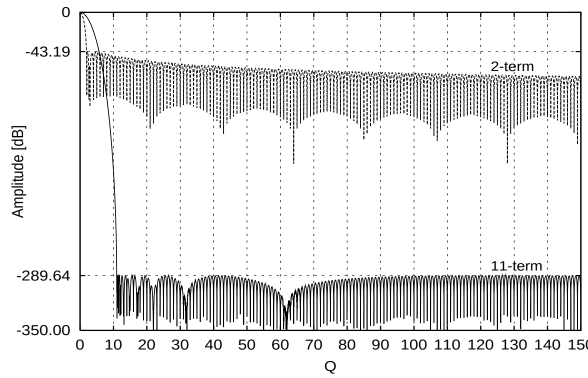

Fig. 2. FFT plot of the 2-term (dotted line) and 11-term minimum sidelobe windows

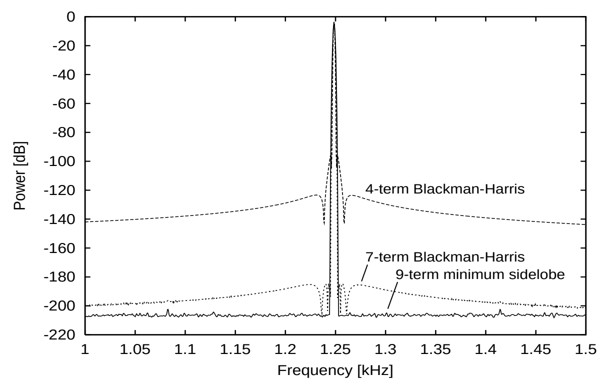

Fig. 3. Output spectrum of a simulated 24 bit ADC with various window functions

Selected parameters of the window functions have been compiled in Table 2. The parameters highest sidelobe level, scalloping loss and bandwidth were determined using equations (13), (6) and (7). Coherent gain and equivalent noise bandwidth (ENBW) were calculated from the coefficients A_p.

Part of the power spectrum of the output signal of a simulated 24 bit ADC at full-scale triggering with a sine signal and a sample frequency of 1 MHz is represented in Fig. 3. The averaged power spectrum ($N = 100$) has been calculated from data sets with 2^{21} samples each. The 9-term minimum sidelobe window (solid line) allows the noise spectrum of the ADC to be observed up to close to the sine signal. Under the given conditions, this is not possible with the minimum 4-term Blackman-Harris window [1] (broken line) or the 7-term Blackman-Harris window [3] (dotted line).

5. SUMMARY

Cosine-sum windows simplify the calculation of the weighting coefficients for different window lengths. The ampli-

Table 1. Coefficients of minimum sidelobe windows

coeff.	2-term window	3-term window
A_0	5.383553946707251e-001	4.243800934609435e-001
A_1	4.616446053292749e-001	4.973406350967378e-001
A_2		7.827927144231873e-002

coeff.	4-term window	5-term window	6-term window	7-term window
A_0	3.635819267707608e-001	3.232153788877343e-001	2.935578950102797e-001	2.712203605850388e-001
A_1	4.891774371450171e-001	4.714921439576260e-001	4.519357723474506e-001	4.334446123274422e-001
A_2	1.365995139786921e-001	1.755341299601972e-001	2.014164714263962e-001	2.180041228929303e-001
A_3	1.064112210553003e-002	2.849699010614994e-002	4.792610922105837e-002	6.578534329560609e-002
A_4		1.261357088292677e-003	5.026196426859393e-003	1.076186730534183e-002
A_5			1.375555679558877e-004	7.700127105808265e-004
A_6				1.368088305992921e-005

coeff.	8-term window	9-term window	10-term window	11-term window
A_0	2.533176817029088e-001	2.384331152777942e-001	2.257345387130214e-001	2.151527506679809e-001
A_1	4.163269305810218e-001	4.005545348643820e-001	3.860122949150963e-001	3.731348357785249e-001
A_2	2.288396213719708e-001	2.358242530472107e-001	2.401294214106057e-001	2.424243358446660e-001
A_3	8.157508425925879e-002	9.527918858383112e-002	1.070542338664613e-001	1.166907592689211e-001
A_4	1.773592450349622e-002	2.537395516617152e-002	3.325916184016952e-002	4.077422105878731e-002
A_5	2.096702749032688e-003	4.152432907505835e-003	6.873374952321475e-003	1.000904500852923e-002
A_6	1.067741302205525e-004	3.685604163298180e-004	8.751673238035159e-004	1.639806917362033e-003
A_7	1.280702090361482e-006	1.384355593917030e-005	6.008598932721187e-005	1.651660820997142e-004
A_8		1.161808358932861e-007	1.710716472110202e-006	8.884663168541479e-006
A_9			1.027272130265191e-008	1.938617116029048e-007
A_{10}				8.482485599330470e-010

Table 2. Parameters of minimum sidelobe windows

Window	Highest Sidelobe Level in dB	Coherent Gain in dB	Scallop Loss in dB	ENBW in Bin	3.0-dB Bandwidth in Bin	6.0-dB Bandwidth in Bin
2-term	43.187	5.37862	1.73868	1.36766	1.30550	1.81884
3-term	71.482	7.44490	1.13525	1.70371	1.61612	2.26377
4-term	98.173	8.78795	0.85056	1.97611	1.86875	2.62431
5-term	125.427	9.81016	0.68006	2.21535	2.09137	2.94118
6-term	153.566	10.64612	0.56526	2.43390	2.29514	3.23077
7-term	180.468	11.33355	0.48523	2.63025	2.47830	3.49095
8-term	207.512	11.92669	0.42506	2.81292	2.64883	3.73304
9-term	234.734	12.45267	0.37780	2.98588	2.81041	3.96231
10-term	262.871	12.92804	0.33950	3.15168	2.96538	4.18209
11-term	289.635	13.34506	0.30908	3.30480	3.10851	4.38506

tude of the sidelobes can be minimized by an appropriate selection of the zeros of the Fourier-transformed window function. A simple optimization method has been used to determine the zeros. The coefficients of the window function are calculated from the zeros without solving a system of linear equations. The coefficients and selected parameters for windows with a peak sidelobe level between 43 dB and 289 dB are stated.

6. REFERENCES

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