JOINT BLIND EQUALIZATION AND ESTIMATION OF THE SYMBOL PERIOD: A CONTRAST FUNCTION APPROACH.

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ABSTRACT

The blind estimation of the symbol period of an unknown linearly modulated signal is addressed. We propose an original methodology relying on the concept of blind deconvolution, which does not suffer from the drawbacks of the classical approaches relying upon the cyclostatinarity of the received signal. Under suitable technical conditions, we show that the optimization of certain cost functions leads to the identification of the symbol rate. Simulation results illustrate the excellent performance of the method.

1. INTRODUCTION AND NOTATIONS

Let $y_a(t)$ denote the complex envelope of the continuous-time signal transmitted by communication system that employs a linear modulation. Signal $y_a(t)$ can be written as $y_a(t) = \sum_k s_k h_a(t - kT_s)$, where the symbol sequence (s_n) is assumed to be i.i.d., where T_s represents the symbol period, and where $h_a(t)$ is the impulse response of an unknown finite impulse response filter resulting from the multi-path channel and from the shaping filter.

A key problem consists in recovering the series (s_n) from the sole knowledge of certain statistics of $y_a(t)$. The related approaches are called "blind" since they do not require any training sequence. In the literature, many papers address the blind equalization problem *when the symbol period T_s is known*. In particular, the deconvolution approach is now very popular: after sampling at the rate $1/T_s$, the (stationary) series is passed through a discrete time filter - say $G(z)$; this one is adapted in order to let its output series "as independent as possible" making the probability distribution of the output $z_n = [G(z)]y_n$ as close as possible from the distribution of (s_n) ([1]). This adaptation step has been shown to reduce to the optimization of certain cost functions, for example the CMA cost function [2], the Maximization of Kurtosis (MK) [3], ... *When T_s is unknown*, the blind equalization problem is clearly much harder. Such a blind problem arises in passive listening for instance. A solution consists in first estimating T_s, then applying the above-mentioned deconvolution technique. The key point is therefore the estimation of T_s. A classical method relies on the fact that $y_a(t)$ is wide-sense cyclo-stationary, with the multiples of $1/T_s$ as cyclo-frequencies. An estimate of T_s can be constructed as the argument maximum of a certain periodogram-like cost function : see [4] for the basic approach,[5][6] for a more elaborate solution. This approach suffers from a clear drawback: generally, the support of the Fourier Transform of the channel h_a is an interval $[-\frac{1+\gamma}{2T_s}, \frac{1+\gamma}{2T_s}]$ where $\gamma \geq 0$ is called the excess bandwidth factor. As is well known, $y_a(t)$ is stationary when $\gamma = 0$. Therefore, the closer γ to 0 (i.e. the minimum bandwidth ensuring a correct transmission [7]) the "more stationary" y_a, hence making the cyclic approach prone to numerical problems [6].

In this paper, we propose a novel approach which rather relies upon the concept of deconvolution. In this respect, let us consider a given sampling period T_e and the associated discrete-time signal $y_a(nT_e)$. We denote by α the ratio $\alpha = \frac{T_e}{T_s}$ and set $y_{n,\alpha} = y_a(n\alpha T_s)$. The series $(y_{n,\alpha})$ is non stationary, but cyclostationary (or almost periodically correlated, APC). We choose to pass $(y_{n,\alpha})$ through a stable digital filter $G(z)$. The filtered time series is denoted by $(z_{n,\alpha})$. In this contribution, we build a family of cost functions $J(G, \alpha)$ which verify the following property: if α is not an integer (i.e. T_e is not a multiple of T_s) then J is strictly beyond a certain specified bound; if $\alpha = 1$, this bound is reached for G a scaled/delayed inverse of the channel. Clearly, such a cost function allows to estimate α as $\text{argmin}_\alpha\ (\min_G J(G, \alpha))$.

We show that such a cost function may derive from the contrast functions used in a stationary context (such as CMA, MK, ...). We provide a sketch of the proof, the full demonstration is out of the scope of this paper and is a part of a full forthcoming paper. We underline that the approach is theoretically robust to a lack of excess bandwidth. Simulations corroborate this remark : In a difficult scenario, when the usual cyclo-correlation based approach is bound to fail (i.e. $\gamma = 0.1$, number N of available symbols $N = 500$), we show that T_s is reliably recovered.

2. CONSTRUCTION OF COST FUNCTIONS

In order to understand the construction of our cost function, we briefly recall the main points of the simple case when $\alpha = 1$.

2.1. When T_s is known (i.e. $\alpha = 1$)

The series $y_{n,1}$ is simply denoted by y_n: it is stationary since it is the convolution $y_n = \sum_k h_k s_{n-k}$ where $h_k = h_a(kT_s)$. We set $H(z) = \sum_k h_k z^{-k}$. Suppose that $1/H(z)$ is stable, a condition which is assumed throughout this paper. Many blind deconvolution/equalization methods consist in adapting a filter $G(z)$ making the probability distribution of the output $z_n = [G(z)]y_n$ as close as possible from the distribution of (s_n) ([1]). As was shown subsequently, one may focus on partial statistics of (z_n): specifically, the minimization of a cost function of the type

$$J_s(G) = \frac{\phi_1(\mathbb{E}\{\psi_1(z_n)\})}{\phi_2(\mathbb{E}\{\psi_2(z_n)\})} \qquad (1)$$

(the suffix $._s$ stands for "stationary") for certain choices of the mappings $\phi_i : \mathbb{R} \to \mathbb{R}$ and $\psi_i : \mathbb{C} \to \mathbb{R}$ achieves the equalization. We use the following terminology

Definition 1 *$G \mapsto J_s(G)$ is a contrast if* **1)** *$J_s(G)$ is uniformly lower-bounded, i.e. if there exists a real constant κ such that, for all stable filter G, $J_s(G) \geq \kappa$* **2)** *the equality occurs when and only when G is scaled/delayed inverse of H.*

Many of the contrast functions are of the type (1):

- when (s_n) has a negative fourth-order cumulant, the MK approach consists in minimizing (1) with $\phi_1(x) = x$, $\psi_1(x) = |x|^4$, $\phi_2(x) = x^2$ and $\psi_2(x) = |x|^2$. Moreover, $\kappa = \kappa_4(s) + 3 = E(s^4)$, where $\kappa_4(s)$ is the fourth-order cumulant of the source.
- when the symbols have constant modulus, the CMA contrast function is of the type (1) with $\phi_1(x) = x$, $\psi_1(x) = (|x|^2 - 1)^2$, $\phi_2(x) = 1$ and $\psi_2 = 1$. The lower bound is $\kappa = 0$.

Notice that in these examples ϕ_1 is concave whereas ϕ_2 is convex. In the sequel, we will restrict to quadruples $(\phi_1, \phi_2, \psi_1, \psi_2)$ verifying:

Assumption 1 *$(\phi_1, \phi_2, \psi_1, \psi_2)$ are such that the associated J_s of Eq. (1) is a contrast function of lower bound κ.*
ϕ_1 (respectively ϕ_2) is concave (respectively convex).

2.2. Generalization: case of an unknown T_s

α is no more bound to be one. The series of interest is now $(y_{n,\alpha})$. Basically, we extend the methodology depicted in a stationary context: the series $(y_{n,\alpha})$ is passed through a digital filter $G(z)$; the resulting series is denoted by $(z_{n,\alpha})$. In general, this series is not stationary but APC. Nevertheless, as in the stationary context, we focus on the inferior bound of a cost function involving certain statistics of $(z_{n,\alpha})$; as is now specified, this function is a direct generalization of the contrast function J_s introduced in Section 2.1. Consider indeed mappings $(\phi_1, \phi_2, \psi_1, \psi_2)$ as depicted in Assumption 1). As $(z_{n,\alpha})$ is APC, $(\mathbb{E}\{\psi_i(z_{n,\alpha})\})_n$ are a priori Almost Periodic deterministic series; we consider now the zeroth-order Fourier coefficient, namely $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}\{\psi_i(z_{n,\alpha})\}$, and define the cost function J associated with J_s (i.e. $J(G,1) = J_s(G)$) :

$$J(G,\alpha) = \frac{\phi_1\left(\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}\{\psi_1(z_{n,\alpha})\}\right)}{\phi_2\left(\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}\{\psi_2(z_{n,\alpha})\}\right)} \quad (2)$$

We now address the minimization of J over the variables G and α.

3. ANALYSIS OF $J(G, \alpha)$

The channel h_a is assumed band-limited, namely we consider the following (technical) restrictions on h_a:

Assumption 2 *$h_a \in L^1(\mathbb{R})$ and denote by $\hat{h}_a(\nu)$ the Fourier Transform of $h_a(t)$ then: $h_a(t) = \int_{-B}^{B} \hat{h}_a(\nu) e^{i2\pi\nu t} d\nu$ where $B = \frac{1+\gamma}{2T_s}$ with $0 < \gamma < 1$*

The main contribution lies in the following

Theorem 1 *Recall that κ is the inferior bound of $J_s(G) = J(G,1)$, then under assumption 1 and 2,*

- *$J(G,\alpha)$ is beyond the bound κ, whatever G and α*
- *a necessary condition for the bound κ to be attained is that α is a non-null integer*
- *the bound is attained for $\alpha_* = 1$ and $G_*(z) = 1/H(z)$.*

Due to the lack of space, we only provide the main steps of the proof in subsections 3.1 and 3.2.

As a preliminary result, it is crucial to see the series $(z_{n,\alpha})$ as the sampled version of a certain continuous-time signal $(z_a(t))$ with sample duration $T_e = \alpha T_s$. In other words, it is possible to prove that for any G there exists a certain filter of impulse response $g_a \in L^2(\mathbb{R})$ such that $z_{n,\alpha} = z_a(nT_e)$ with $z_a(t) = \sum_n s_n f_a(t - nT_s)$ and $f_a(t) = g_a \star h_a(t)$. Among these solutions, we pick up the following

$$D(G,\alpha)(t) = \sum_k g_k \text{sinc}(4\pi B(t - k\alpha T_s)). \quad (3)$$

We naturally define the functional in g_a (the existence is discussed in Lemma 1).

$$\tilde{J}(g_a,\alpha) = \frac{\phi_1\left(\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}\{\psi_1(z_a(n\alpha T_s))\}\right)}{\phi_2\left(\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}\{\psi_2(z_a(n\alpha T_s))\}\right)} \quad (4)$$

such that we have the following relation between J and $\tilde{J}$: $J(G,\alpha) = \tilde{J}(D(G,\alpha),\alpha)$. In other words, this relation legitimates that we concentrate on $\tilde{J}$.

3.1. A basic property of $\tilde{J}$

For a given g_a, the process $z_a(t)$ is cyclostationary with period T_s. Hence $\mathbb{E}\{\psi_i(z_a(t))\}$ is a T_s-periodic mapping. We make the technical assumption that $t \mapsto \mathbb{E}\{\psi_i(z_a(t))\}$ is continuous. We can define the k^{th}-order Fourier coefficient denoted by $\Psi_i^{(k)}$:

$$\Psi_i^{(k)}(g_a) = \frac{1}{T_s}\int_0^{T_s} \mathbb{E}\{\psi_i(z_a(t))\} \exp\left(-i2\pi k \frac{t}{T_s}\right) dt. \quad (5)$$

We have

Lemma 1 *Assume that for $i = 1, 2$ the series $\left(\Psi_i^{(k)}(g_a)\right)_k$ are summable. If $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, then*

$$\tilde{J}(g_a,\alpha) = \frac{\phi_1\left(\Psi_1^{(0)}(g_a)\right)}{\phi_2\left(\Psi_2^{(0)}(g_a)\right)}; \quad (6)$$

If $\alpha = \frac{p}{q}$ with p, q coprime integers, then

$$\tilde{J}(g_a,\alpha) = \frac{\phi_1\left(\sum_k \Psi_1^{(kq)}(g_a)\right)}{\phi_2\left(\sum_k \Psi_2^{(kq)}(g_a)\right)}. \quad (7)$$

This preliminary result calls for comments. Lemma 1 thus implies that, in general, for a fixed g_a, the mapping $\alpha \mapsto \tilde{J}(g_a,\alpha)$ is discontinuous in every α. Moreover, if α is irrational, it is worth noticing that $\tilde{J}(g_a,\alpha)$ does not depend on α (see (6)).

3.2. Lower bound of $\tilde{J}$

Lemma 2 *Suppose that the technical condition of Lemma 1 holds and that Assumptions 1 (relative to the considered cost function) and 2 (relative to h_a) are fulfilled. For any non-integer α and any $g_a \in L^2(\mathbb{R})$, we have $\tilde{J}(g_a,\alpha) > \kappa$.*

As is suggested by Lemma 1, the behavior of $\tilde{J}$ depends essentially on the rational character of α; the two cases (α rational and α irrational) must be treated separately. The proof of this lemma takes this remark into account. Due to the lack of space, we consider the case α irrational. The other case can be treated the same way. We may express $\tilde{J}$ as in (6), with the $\Psi_i^{(k)}(g_a)$'s given by (5). As ϕ_1 is a concave mapping, it yields from Jensen's inequality

$$\phi_1\left(\frac{1}{T_s}\int_0^{T_s}\mathbb{E}\{\psi_1(z_a(t))\}dt\right) \geq \frac{1}{T_s}\int_0^{T_s}\phi_1\left(\mathbb{E}\{\psi_1(z_a(t))\}\right)dt. \tag{8}$$

For a *given* real t, $z_a(t) = \sum_k s_k f_a(t - kT_s)$ and can be seen as a discrete-time convolution. As the cost function J_s in (1) associated with the mappings $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ is a contrast function, it yields

$$\phi_1\left(\mathbb{E}\{\psi_1(z_a(t))\}\right) \geq \kappa\phi_2\left(\mathbb{E}\{\psi_2(z_a(t))\}\right) \tag{9}$$

for each t. Hence

$$\frac{1}{T_s}\int_0^{T_s}\phi_1\left(\mathbb{E}\{\psi_1(z_a(t))\}\right)dt \geq \kappa\frac{1}{T_s}\int_0^{T_s}\phi_2\left(\mathbb{E}\{\psi_2(z_a(t))\}\right)dt$$
$$\geq \kappa\phi_2\left(\frac{1}{T_s}\int_0^{T_s}\mathbb{E}\{\psi_2(z_a(t))\}dt\right)$$

the last inequality coming from the Jensen's formula applied to the convex mapping ϕ_2; this proves that $\tilde{J}(g_a, \alpha) \geq \kappa$ whatever $g_a \in L^2(\mathbb{R})$ and α irrational. We notice that a necessary condition for the inequality to be an equality is that (9) is an equality for almost every $t \in [0, T_s]$: this gives the following condition on f_a:

$$f_a(t + nT_s) = \lambda(t)\delta_{n-\tau(t)} \tag{10}$$

where $\lambda(t)$ are non-null constants and $\tau(t)$ integers. The following result can be easily proven: due to bandwidth limitation (see Assumption 2), it is impossible to find a g_a such that $g_a \star h_a$ verifies (10). Hence we eventually have $\tilde{J}(g_a, \alpha) > \kappa$. This concludes the proof when α is irrational.

3.3. Minimization of J

We now focus on the following quantity: $\inf_G J(G, \alpha)$. Suppose that α is not an integer. At a first glance, Theorem 1 seems to provide that $\inf_G J(G, \alpha) > \kappa$. This is of course not obvious. This result can be shown to be true as soon as a certain mathematical specifications are provided (G a compact subset of $l^1(\mathbb{Z})$, continuity of $g_a \mapsto \tilde{J}(g_a, \alpha)$ with respect to a certain norm ...); this task is done in a forthcoming full paper. Actually, it is possible to show the following stronger result:

Theorem 2 *if the ψ_i's are polynomial mappings, there exists a constant κ' such that for any non integer α:*

$$\inf_G J(G, \alpha) \geq \kappa' > \kappa.$$

This uniform bound in α has an immediate importance on a statistical point of view, since the mapping $\alpha \mapsto \inf_G J(G, \alpha)$ has a discontinuity for $\alpha = 1$ (and also for the other integers), which makes the detection of T_s easy. Of course, the bigger the gap $\kappa' - \kappa$, the more robust the detection of T_s; the theory does not give any specification on $\kappa' - \kappa$; however, simulation results illustrate that $\kappa' - \kappa$ is enough to ensure a correct detection of α.

Remark 1 *It can be shown that the gap $\kappa' - \kappa$ is all the bigger as the channel has a small roll-off. This implies the less the roll-off the better the performances. This surprising result is confirmed by simulation.*

4. EXAMPLE AND ALGORITHM

We now present the practical settings of our method with an example: the MK cost function. We use the classical empirical averaging to estimate J. Namely we consider:

$$\hat{J}(G, \alpha) = \frac{\frac{1}{N}\sum_{n=0}^{N-1}|z_{n,\alpha}|^4}{(\frac{1}{N}\sum_{n=0}^{N-1}|z_{n,\alpha}|^2)^2} \tag{11}$$

As N is finite, this function is continuous and to estimate T_s, we have to minimize $\hat{J}(G, \alpha)$ on G and α. It is not possible to obtain a tractable form of the estimated cost function as a function of α. Thus, we choose a grid of α which corresponds to the search area of T_s. And for each α in this grid we compute the value of (11) by minimizing on G the cost function using a newton algorithm. We estimate α as the argument minimum of all the values previously computed. The choice of the step of the grid is then crucial and discussed in section 5.

It is quite easy to have an rough idea of the bandwidth of the received signal and thus we choose a initial sampling period T_i which verify the Shannon sampling condition. We sample the received signal at T_i, and get the sequence $\{y_a(nT_i)\}$. Thanks to this initial sequence, we are able by the Shannon's interpolation theorem to generate any sequence $\{y_a(n\alpha T_s)\}$ for any α.

5. SIMULATION RESULTS

We now illustrate the good performance of our approach, we first study the impact of the observation duration (i.e. NT_s) on the setting of the grid of α, then we illustrate the influence of the roll-off on our contrast-function. Finally, we compare our approach to the cyclo-correlation based approach.

We first present the context used for the different simulations. The emitted signal always originates from a digital source modulated by a PSK4 i.i.d sequence shaped by a square root Nyquist filter with roll-off γ. We consider a multi-path channel with 3 paths. The delays of the paths are respectively : $(0.40046T_s; 2.08262T_s; 2.96008T)$ and the complex amplitudes are : $(0.89 + 0.43i; -0.50 - 0.14i; 0.21 + 0.24i)$. This channel is used for all the following simulations. The two first simulations are run in a noise free context. In applying the CMA and MK algorithms, we have always used a 4-tap equalizer. Theoretically the inverse of $H(z)$ should be an IIR filter, but numerically, this inverse is well approximated by a 4-tap equalizer. The signal $y_a(t)$ is modeled as a baseband output of 1 antenna. As shown in section 4, the choise of the grid is very crucial.

Figure 1 illustrates the influence of the observation duration NT_s on $\hat{J}(G, \alpha)$ around $\alpha = 1$: The more samples we use, the sharper the peak which means the better the estimation. But in order to detect a very sharp peak, we need a very dense grid of α.

Let now examine the influence of the roll-off, we set $N = 1000$. We run 10 simulations with different symbols sequence and plot in figure 2 the average of $\mathrm{Min}_G(\hat{J}(G, \alpha))$ for two different roll-off values: $\gamma = 0.1$ and $\gamma = 0.7$. The minimum is reached for $\alpha = 1$ and is equal to 0. For $\alpha = k$, $k > 1$, the equalizer $g(z)$ can not equalize perfectly and the minimum is not reached. Indeed when $\alpha = k$, $k > 1$, the SISO system is equivalent to a MISO system with k sources. In figure 2, we establish that the roll-off has an influence on the value of the cost function when α is far away from 1. Thanks to this numerical illustration, we presume on the accuracy of the estimation of T_s even for small roll-off.

The less the roll-off the better the detection . Indeed the gap between the minimum of the cost function and the value of the cost function when T_e is far away from T_s is greater for small roll-off which means that the detection is easier. In a full band context (i.e. $\gamma = 1$), there exists an equalizer for $\alpha = k/2$, $k \in \mathbb{Z}$ and in this case, the minimum will be also reached for $\alpha = k/2$. This explains the point measured for a roll-off equal to 0.7 at $\alpha = 1/2$.

We now focus on the comparison between our method and the classical cyclostationary based approach. Basically this latter consists in : $\hat{\alpha} = \text{ArgMin}_{\alpha,\alpha\neq 0}\hat{R}^{(\alpha)*}W\hat{R}^{(\alpha)}$ where W is a well choosen weighting matrix and $\hat{R}^{(\alpha)} = (\hat{R}^{(\alpha)}(-n), \ldots, \hat{R}^{(\alpha)}(n))^T$ a $(2n+1)$ column vector where $\hat{R}^{(\alpha)}(\tau)$ is the empirical estimate of cyclic correlation at frequency α and delay τ of a sampled version of $y_a(t)$, the sampling period T_c must verify the Shannon condition. In practice, the minimization versus α is achieved by an exhaustive search over a grid with a step equal to $\frac{T_c}{NT_s}$. We study the probability of detection of the peak for both methods. We assumed to have a correct detection when $|\hat{T}_s - T_s| < T_s/N$. We compare the probabilities of correct detection versus the roll-off and versus the signal to noise ratio (SNR). We test 2 cyclic methods [6]: the classical one in which $W = I$ (noted "cy") and the weighted one in which W is choosen in a optimal way [5][6] (noted "cyw"). We compare them to our MK and CMA cost functions. We test 2 different noise levels: 10 and 30 dB, 2 different roll-offs: 0.2 and 0.7, and for $N = 500$. We assume the noise Gaussian. We run 100 realizations and count the number of correct detection. The results are presented in table 1. The bad detection for the mk and cma based approach at

	$\gamma = 0.2$				$\gamma = 0.7$			
SNR	cy	cyw	mk	cma	cy	cyw	mk	cma
10 dB	07	43	100	98	90	100	58	53
30 dB	12	53	100	100	99	100	100	100

Table 1. Number of correct detections over 100 realizations

SNR=10dB and for a roll-off of $\gamma = 0.7$ lies on two facts: first at small SNR and high roll-off, it is possible to confound the peak at $\alpha = 1/2$ and the one at $\alpha = 1$. Secondly, for large value of α, the number of samples used to estimate the cost function is smaller than for small α (i.e the observation duration stay constant), which means that the variance of the estimation of our cost function is greater for large α. Thus at small SNR, we will not be able to detect the peak correctly.

6. CONCLUSION

In this contribution, we develop a methodolodgy for the estimation of the symbol period. It relies on the idea of deconvolution. We set forth some fonctions the minimization of which theoretically provides the symbol period. Moreover, the approach has been validated by extensive simulations which confirm its superiority over the existing methods when the excess bandwidth is tiny.

7. REFERENCES

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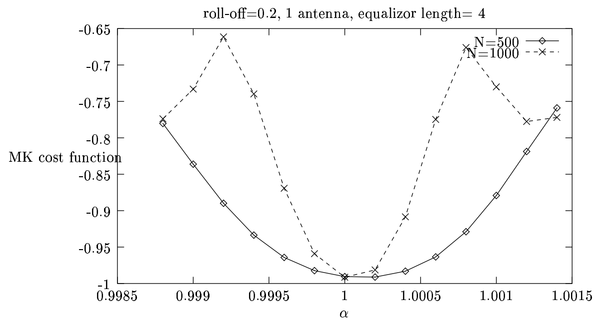

Fig. 1. Zoom on the MK cost function in a neighborhood of $\alpha = 1$, $N = 500$ and $N = 1000$

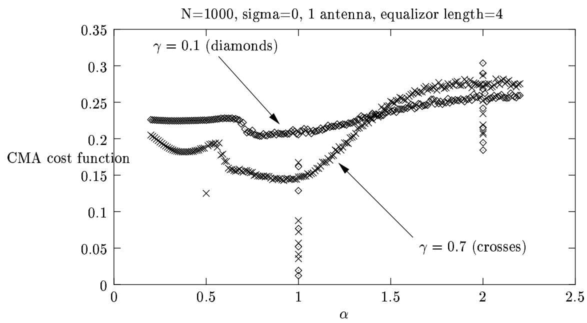

Fig. 2. Influence of the roll-off on the CMA cost function