DECIMATION AND SVD TO ESTIMATE EXPONENTIALLY DAMPED SINUSOIDS IN THE PRESENCE OF NOISE

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ABSTRACT

A new state-space method for spectral estimation that performs decimation by factor two while it makes use of the full set of data available is presented in this paper. The proposed method, called DESE2, is based on Singular Value Decomposition in order to estimate frequency, damping factor, amplitude and phase of exponentially damped sinusoids in the presence of noise. The DESE2 method is compared against some previously proposed methods for spectral estimation that lie among the most promising methods in the field of spectroscopy, where accuracy of parameter estimation is of utmost importance. Experiments performed on a typical simulated NMR signal prove the new method to be more robust, especially for low signal to noise ratio. The new method outperforms the other two not only by presenting lower failure rates but also by incorporating enhanced discriminative analysis while at the same time it benefits from the use of the full data set.

1. INTRODUCTION

Exponential sinusoidal models, often employed in order to represent a signal segment as a sum of exponentially damped complex-valued sinusoids ([5]), are used in various applications of digital signal processing, including speech processing [4] as well as spectroscopy, i.e. quantification of NMR signals. The generalised sinusoidal model we use is given by

$$\begin{aligned} s(n) &= \sum_{i=1}^{p} (b_i e^{j(\phi_0+\phi_i)}) e^{(-d_i+j2\pi f_i)n} \\ &= \sum_{i=1}^{p} g_i\, z_i^n, n=0,...,N-1 \end{aligned} \quad (1)$$

where p is the model order, i.e. the number of sinusoids that comprise the measured signal. The objective is to estimate the frequencies f_i, damping factors d_i, amplitudes b_i and phases $\phi_0+\phi_i$, $i=1,\cdots,p$. ϕ_0 is the zero order phase, whereas ϕ_i represents extra degrees of freedom.

The new method proposed here is called DESE2 (DEsimative Spectral Estimation2), performs decimation by factor two while it exploits the full data set and makes use of decimated Hankel derived matrices and Singular Value Decomposition. DESE2 has been tested and compared to HTLS [3], the latter being one of the most promising methods for parameter estimation which is based on the use of total least squares. Moreover, DESE2 has been tested and compared to MATPEN [2], that also makes use of a Hankel matrix and SVD. Description of the proposed method follows and the superior performance of DESE2 is shown through Monte-Carlo based experiments.

2. THE DESE2 METHOD

2.1. Description and Proof

Let S_H be the $L \times M$ Hankel signal observation matrix of our deterministic signal of p exponentials $s(n)$.

$$S_H = \begin{pmatrix} s(0) & s(1) & \cdots & s(M-1) \\ s(1) & s(2) & \cdots & s(M) \\ \vdots & \vdots & & \vdots \\ s(L-1) & s(L) & \cdots & s(N-1) \end{pmatrix} \quad (2)$$

with $L-2<M$, $p<L-2$ and $L+M-1=N$.

Let $\tilde{s}_n$ be the column vectors of S_H, i.e. $S_H = [\tilde{s}_0 \tilde{s}_1 \cdots \tilde{s}_{M-1}]$, where $\tilde{s}_n = [s_n s_{n+1} \cdots s_{L+n-1}]^T$, for $n=0,1,\cdots,M-1$.

We consider the $L \times K$ matrix S formed as a column rearrangement of the Hankel S_H, where all column vectors $\tilde{s}_n$ with even indices are moved to the left part of the matrix, while the odd column vectors are moved to the right part, R=*floor*$((M-2)/2)$ and $K=2(R+1)$.

$$S = [\tilde{s}_0 \tilde{s}_2 \cdots \tilde{s}_{2R} \vdots \tilde{s}_1 \tilde{s}_3 \cdots \tilde{s}_{2R+1}] \quad (3)$$

Let the $L-2\times K$ matrix $S\downarrow\downarrow$ be the second order lower shift (top two rows deleted) equivalent of S and $S\uparrow\uparrow$ be the second order upper shift (bottom two rows deleted) equivalent of S.

$$
\begin{aligned}
S\downarrow\downarrow &= [\hat{s}_2\hat{s}_4\cdots\hat{s}_{2R+2}\vdots\hat{s}_3\hat{s}_5\cdots\hat{s}_{2R+3}] \\
S\uparrow\uparrow &= [\hat{s}_0\hat{s}_2\cdots\hat{s}_{2R}\vdots\hat{s}_1\hat{s}_3\cdots\hat{s}_{2R+1}]
\end{aligned} \tag{4}
$$

where column vectors $\hat{s}_n$ (of length $L-2$), for $n=0,1,\cdots,M-1$ are the column vectors $\tilde{s}_n$ with their two last components removed.

Because of the fact that S_H is rank deficient and any row of $S\downarrow\downarrow$ can be written as a linear combination of the rows of $S\uparrow\uparrow$, there is an (L -2) order matrix X, such that,

$$XS\uparrow\uparrow = S\downarrow\downarrow \tag{5}$$

We will now prove that all the signal's poles are contained in the eigenvalues of X. Since $\hat{s}_n$ are the column vectors of $S\uparrow\uparrow$, Eq.(5) can be expressed as

$$
\begin{aligned}
X\hat{s}_0 &= \hat{s}_2 \\
X\hat{s}_2 &= \hat{s}_4 \\
&\vdots \\
X\hat{s}_{2R} &= \hat{s}_{2R+2} \\
X\hat{s}_1 &= \hat{s}_3 \\
X\hat{s}_3 &= \hat{s}_5 \\
&\vdots \\
X\hat{s}_{2R+1} &= \hat{s}_{2R+3}
\end{aligned} \tag{6}
$$

which can be written as $\hat{s}_n = X^k\hat{s}_0$, if $n=2k$ and $\hat{s}_n = X^k\hat{s}_1$, if $n=2k+1$ $\forall n, n=0,1,\cdots,2R+3$.

Note that we can also deduce the set of equations in (6), which is derived from (5), if we consider an alternative rearrangement of matrix S_H. Let the $L-2\times M$ matrix $S{\downarrow}_2$ be the second order lower shift (top two rows deleted) equivalent of S_H and $S{\uparrow}_2$ be the second order upper shift (bottom two rows deleted) equivalent of S_H. The set deriving from equation $XS{\uparrow}_2 = S{\downarrow}_2$ is identical to (6).

In general, matrix X can be diagonalised as follows

$$X = U\Lambda U^{-1} \tag{7}$$

thus, the set of equations in (6), can be expressed as $\hat{s}_n = U\Lambda^k U^{-1}\hat{s}_0$, if $n=2k$ and $\hat{s}_n = U\Lambda^k U^{-1}\hat{s}_1$, if $n=2k+1$ $\forall n, n=0,1,\cdots,2R+3$.

The above implies that signal s can be written as a linear combination of the eigenvalues of matrix X, i.e. $s_n = \sum_{j=1}^{L-2} c_j{\lambda_j}^n$, $\forall n$, if the decimated poles are converted to signal poles by appropriate adjustment.

Note that all λ_j's correspond to the signal. However, by definition, the signal s consists of p sinusoids and is expressed as a function of its poles z's: $s_n = \sum_{k=1}^{p} g_k\, z_k^n$, $\forall n$.

Given that the signal can be uniquely expressed as a function of its exponentials, it is now easily deduced that p of the λ_j's correspond to the signal's poles (the rest of the λ_j's are associated to zero gain c_j).

Finally we assign $X = S\downarrow\downarrow pinv(S\uparrow\uparrow)$, which satisfies Eq.(5), and we deduce the decimated poles of the signal s by computing the eigenvalues of X.

In the presence of noise, which is the case of real life signals, the rank of matrix S is full. Moreover, since the signal does not obey linear models the equality (5) does not hold any longer. In such cases, matrix $S\uparrow\uparrow$ can be enhanced by reducing its rank to p (p being the number of complex peaks to estimate). For that purpose we employ the SVD of $S\uparrow\uparrow$ and we retain the p largest singular values. The resulting matrix $S\uparrow\uparrow_e$ has rank p. Then X is computed from $XS\uparrow\uparrow_e\approx S\downarrow\downarrow$ which gives rise to an overdetermined system of equations with the following solution $X = S\downarrow\downarrow pinv(S\uparrow\uparrow_e)$.

Since matrix $S\uparrow\uparrow_e$ has rank p, X is also of rank p. Hence only p of the eigenvalues of X are non-zero and correspond to the decimated signal poles estimates. Thus, the desired decimated estimates of frequencies and damping factors are calculated as the angles and magnitudes respectively of the eigenvalues of X. These decimated estimates are converted to their non decimated equivalents f_i (frequency estimates) and d_i (damping factor estimates) and a computation in a total least squares sense of estimates g_i takes then place. Furthermore, amplitude b_i and phases $\phi_0+\phi_i$ estimates are determined as the magnitudes and angles of g_i respectively.

2.2. Algorithmic Presentation

The proposed algorithm for decimation factor two involves the following five steps:

Step 1: We compute the $L\times M$ matrix S_H of Eq.(2) from the N data points $s(n)$ of Eq.(1).

Step 2: We compute the $S{\downarrow}_2$ and $S{\uparrow}_2$ as the $2nd$ order lower shift (top 2 rows deleted) and the $2nd$ order upper shift (bottom 2 rows deleted) equivalents of S_H. The best results are obtained when we use the $L-2\times M$ matrices $S{\downarrow}_2$ and $S{\uparrow}_2$ as square as possible ([3]).

Step 3: We compute the enhanced version $S{\uparrow}_2e$ of $S{\uparrow}_2$ in the following way: We employ the SVD of $S{\uparrow}_2$, $S{\uparrow}_2 = U{\uparrow}_2\Sigma{\uparrow}_2 V{\uparrow}_2^T$ and we truncate to order p by retaining only the largest p singular values.

Step 4: We compute matrix $X = S{\downarrow}_2 pinv(S{\uparrow}_2 e)$. The eigenvalues $\hat{\lambda}_i$ of X give the decimated signal pole estimates, which in turn give the estimates for the damping factors and frequencies of Eq. (1).

Step 5: The last step is to compute the phases and the amplitudes. This is done by finding a least squares solution to Eq. (1), with z_i replaced by the estimates and $s(n)$ given by the signal data points.

2.3. Special Cases

The above presented method can also serve as a state-space method for spectral estimation, if seen and implemented with no decimation whatsoever. In this case, matrices $S_1\downarrow$ and $S_1\uparrow$ are respectively the first order lower shift (top row deleted) and first order upper shift (bottom row deleted) of the original Hankel S_H of Eq. (2) with $L < M$, $p < L - 1$ and $L + M - 1 = N$.

A variation of such a non decimative method, called CSE, had been proposed in [1]. In this case both matrices $S_1\downarrow$ and $S_1\uparrow$ (of *Step 2*) were enhanced (truncated to order p) with the use of SVD, resulting in matrices $S_1\downarrow_e$ and $S_1\uparrow_e$ respectively. Thus, matrix X of *Step 4* is computed by $X = S_1\downarrow_e pinv(S_1\uparrow_e)$.

If only matrix $S_1\uparrow$ is enhanced, the non decimative method is identical to a method proposed in [2], which we call MATPEN in the discussion that follows and for which calculation of X (in *Step 4*) is achieved by $X = S_1\downarrow pinv(S_1\uparrow_e)$.

3. EXPERIMENTAL RESULTS

All three methods, namely DESE2, MATPEN and HTLS have been tested via simulations on a typical five peak ^{31}P NMR signal of perfused rat liver, in order to evaluate both robustness as well as the improvement in accuracy of parameter estimation when using the three methods in the modelling problem defined by Eq.(1). This ^{31}P NMR signal comprises a fifth-order model function given in Table 1 by which N data points uniformly sampled at 10KHz are exactly modelled. The data points of the signal are perturbed by Gaussian noise whose real and imaginary components have standard deviation σ_v.

peak i	f_i (Hz)	d_i (rad/s)	b_i	$\psi_i^{(a)}$
1	-1379	208	6.1	15
2	-685	256	9.9	15
3	-271	197	6.0	15
4	353	117	2.8	15
5	478	808	17.0	15

$^{(a)}$ $\psi_i = \phi_0 \star 180\pi$ expresses the phase in degrees

Table 1: Exact parameter values of the five peak simulated ^{31}P NMR signal, modelled by Eq.(1) with $\phi_i = 0$.

Root mean squared errors of the estimates of all signal parameters are computed using 500 noise realizations (excluding *failures*) for different noise levels. We consider that a failure occurs when *not all peaks* are resolved within specified intervals lying symmetrically around the exact frequencies. For our signal, the halfwidths of the intervals are respectively 82, 82, 82, 43 and 82 Hz, the values being derived from the Cramer-Rao bounds of peaks 4 and 5 at the noise standard deviation where these intervals are touching each other. The estimated model order is set to 5. The Cramer-Rao lower bounds are derived from the exact parameter values and σ_v.

In Fig. 1 failure rate of the three methods is depicted as a function of noise standard deviation (N=128 and $M = N/2 = 64$). Clearly DESE2 and MATPEN are more robust than HTLS. Comparison between DESE2 and MATPEN proves DESE2 more robust, where a decimative approach is expected to outperform any non decimative one. This is due to the fact that decimation brings peaks further apart, thus, increasing the discriminative capacity of spectral estimation methods.

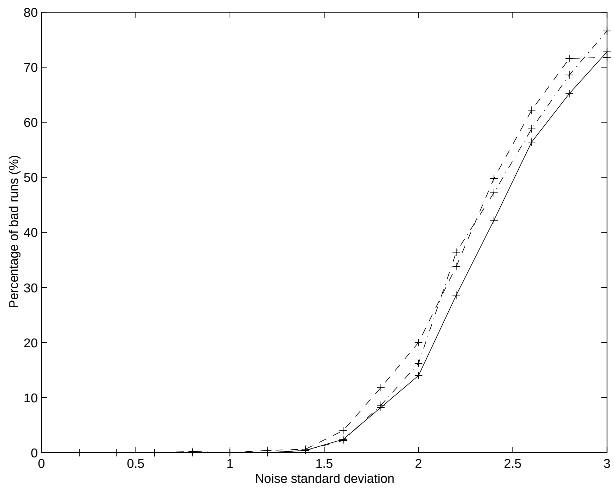

Figure 1: Percentage of times that that DESE2 (solid), MATPEN (dashdot) and HTLS (dashed) fail to resolve all peaks of the five peak simulated ^{31}P NMR signal as a function of noise standard deviation σ_v.

Noise standard deviation as well as root mean squared errors of frequency and damping factor are shown below in tabular format. Table 2 presents the results for peak 1 and peak 5 of the five peak simulated ^{31}P NMR signal. Peak 5 of this signal is considered the most difficult to estimate since it is relatively close to peak 4. The results suggest that the decimative approach performs similarly (to MATPEN and HTLS) for high S/N ratio. However, for low S/N ratio, despite the similarity of the root mean squared errors of all parameters estimated, DESE2 performs better due to its lower failure rate and is, thus, more robust.

Note that there are cases where DESE2 outperforms, in terms of root mean squared errors, HTLS and MATPEN despite the fact that it has smaller number of bad runs. In some cases, however, the results presented in the tables are better for HTLS and MATPEN because they present higher number of bad runs than DESE2.

4. CONCLUSION

A new state-space decimative method, called DESE2, for spectral estimation was presented. The proposed method makes use of decimation factor two and SVD, in order to estimate the parameters (frequencies, damping factors, amplitudes and phases) of exponentially damped complex-valued sinusoids in the presence of noise. DESE2 was tested in spectroscopy, the latter lying among the most demanding applications of digital signal processing in terms of accuracy. DESE2 was compared to two state-of-the-art non decimative methods in spectroscopy, the MATPEN and HTLS methods. Examples on a typical five peak ^{31}P NMR signals were presented and the superior performance of DESE2 over the other methods was shown, especially for low signal to noise ratio.

5. REFERENCES

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Estimates for	Peak 1		Peak 5	
σ_v/Method	f_1 (Hz)	d_1 (rad/s)	f_5 (Hz)	d_5 (rad/s)
0.2/DESE2	0.4931	0.0290	2.0755	0.0193
0.2/MATPEN	0.4945	0.0158	2.0554	0.0174
0.2/HTLS	0.4933	0.0158	2.1295	0.0166
0.4/DESE2	1.0512	0.0335	4.1865	0.0322
0.4/MATPEN	1.0377	0.0333	4.2146	0.0316
0.4/HTLS	1.0379	0.0333	3.9962	0.0437
0.6/DESE2	1.4939	0.0473	6.2757	0.0748
0.6/MATPEN	1.4989	0.0468	6.0667	0.0482
0.6/HTLS	1.4967	0.0465	6.3048	0.0473
0.8/DESE2	2.0393	0.0652	9.1107	0.0759
0.8/MATPEN	2.0261	0.0656	8.9001	0.0647
0.8/HTLS	2.0192	0.0649	9.2478	0.0614
1.0/DESE2	2.7210	0.0789	11.2764	0.1004
1.0/MATPEN	2.7020	0.0789	11.0553	0.0869
1.0/HTLS	2.7323	0.0778	10.6887	0.0818
1.2/DESE2	3.2078	0.0954	14.3753	0.1714
1.2/MATPEN	3.1576	2.6293	14.3304	0.1150
1.2/HTLS	3.1669	0.0939	13.7155	0.0988
1.4/DESE2	3.9644	0.1178	16.7876	0.1324
1.4/MATPEN	3.9646	0.1170	16.9051	0.1263
1.4/HTLS	3.9645	0.1132	15.5581	0.1095
1.6/DESE2	4.3295	0.1328	21.7850	0.1684
1.6/MATPEN	4.3441	0.1331	22.6016	0.1735
1.6/HTLS	4.3390	0.1297	20.1668	0.1480
1.8/DESE2	5.0269	0.1499	26.5238	0.2227
1.8/MATPEN	5.0484	0.1507	28.3658	0.2877
1.8/HTLS	5.0948	0.1457	25.0569	0.1672
2.0/DESE2	5.5988	0.1770	30.3730	0.4047
2.0/MATPEN	5.6167	0.1798	30.5536	0.3695
2.0/HTLS	5.5942	0.1639	27.5575	0.1792
2.2/DESE2	6.1303	0.1974	32.1792	0.3596
2.2/MATPEN	6.1611	0.1955	33.4734	0.4611
2.2/HTLS	6.5434	0.1827	29.1000	0.2130
2.4/DESE2	7.6502	0.2220	35.0694	0.3805
2.4/MATPEN	7.4973	0.2244	34.1181	0.4270
2.4/HTLS	7.5257	0.2050	33.1237	0.2770

Table 2: Root mean squared errors of frequency and damping factor for peak 1 and peak 5 of the five peak simulated ^{31}P NMR signal described in Table 1 as a function of noise standard deviation σ_v. The values below the double (triple) horizontal line correspond to smaller number of bad runs for DESE2 compared to HTLS (MATPEN), whereas the values above to the same number.