INSTANTANEOUS FREQUENCY ESTIMATION BY USING TIME-FREQUENCY DISTRIBUTIONS

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ABSTRACT

Estimation of the instantaneous frequency by using quadratic distributions from the general Cohen class is analyzed. Frequency modulated signals corrupted with a white stationary noise are considered. Expression for the variance is derived. It is shown that the variance is closely related to the non-noisy distribution of a predefined signal.

1. INTRODUCTION

Instantaneous frequency (IF) estimation is an important research topic in signal analysis [2, 3, 5, 6, 7, 9]. There are several approaches to the IF estimation, one of them being based on time-frequency distributions (TFD) [2, 3, 9]. TFDs concentrate the signal's energy at and around the IF in the TF plane [2, 3, 5, 6, 7, 9]. Consequently, peak detection of TFDs' is used as an IF estimator.

The IF estimation based on TFDs maxima is analyzed in [2, 5, 6, 7, 9]. Only the most frequently used TFDs: the Wigner distribution (WD) for linear frequency-modulated (FM) signal, and the spectrogram (SPEC) for signals whose IF could be considered as a constant within the lag window, are presented there. It has been shown that the IF estimate is highly dependent on signal, noise, and lag window length.

In this paper we present a general analysis of an arbitrary shift covariant quadratic TFD as an IF estimator, for any FM signal corrupted by white stationary noise. The exact expression for the IF estimator variance is derived. Expressions for some frequently used TFDs from the Cohen class (CD) are obtained as special cases, as well. Variance for the SPEC as an IF estimator of a linear FM signal is presented. This signal is considered in the cases of other commonly used TFDs, such as the Born-Jordan and Choi-Williams distributions. It has been shown that the reduced interference distributions outperform the WD, but only in the case when the IF is constant or its variations are small. For highly nonstationary signals the WD can produce better IF estimation.

In Section 2, the IF estimator based on quadratic distributions is defined and analyzed. In Section 3 the analysis of the estimation error is performed. The variance of the estimation error, in the cases of commonly used quadratic TFDs, is presented next. The obtained results are checked numerically and statistically in Section 5.

2. BACKGROUND THEORY

Consider discrete-time observations,

$$x(nT) = f(nT) + \epsilon(nT), \;\; f(t) = A\exp(j\phi(t)) \tag{1}$$

where: n is an integer; T is a sampling interval; $\epsilon(nT)$ is a stationary, complex, white, Gaussian noise, with the variance σ_ϵ^2; and A is the amplitude of analyzed signal. By definition, [3, 5, 6, 7], the IF is the first derivative of the signal phase, $\omega(t) = \phi'(t) \equiv d\phi(t)/dt$. It can be estimated from the discrete-time observations (1). Here we assume that $\omega(t)$ is an arbitrary smooth differentiable function of time, with bounded derivatives $\left|\omega^{(r)}(t)\right| = \left|\phi^{(r+1)}(t)\right| \leq M_r(t)$, $r \geq 1$.

General form of the quadratic shift-covariant TFD's in the discrete time domain is given by

$$C_x(t,\omega;\varphi_h) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\varphi_h(mT,nT)\times$$

$$\times x(t+mT+nT)x^*(t+mT-nT)e^{-j2\omega nT} \tag{2}$$

where $\varphi_h(mT,nT) = (T/h)^2\varphi(mT/h,nT/h)$, with symmetric time-lag kernel $\varphi(t,\tau)$. Suppose that $\varphi(t,\tau)$ has a finite length along both directions, $\varphi(t,\tau) = 0$, for $|t| > 1/2$ and $|\tau| > 1/2$. It means that $\varphi_h(mT,nT)$ is limited in both directions by h, $h > 0$. Note that h is used in definition of the CD in order to localize the estimate.

Let us analyze the CD of the signal $f(t)$. Expanding $\phi(t+mT \pm nT)$ into the Taylor series around t (up to the third order term), we get:

$$C_f(t,\omega;\varphi_h) = |A|^2\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\varphi_h(mT,nT)\times$$

$$e^{-j[2(\omega-\phi'(t))(nT)-2\phi^{(2)}(t)(mT)(nT)-\triangle\phi(t,mT,nT)]} \tag{3}$$

where $\triangle\phi(t,mT,nT)$ is the residue of the phase. It may be represented as:

$$\triangle\phi(t,mT,nT) = \sum_{s=3}^{\infty}\frac{\phi^{(s)}(t)}{s!}\sum_{k=0}^{s}\binom{s}{k}(mT)^{s-k}(nT)^k\times$$

$$\times[1-(-1)^k]. \tag{4}$$

Note that TFDs from the CD would have a maximum at $\omega = \phi'(t)$ if $\phi^{(s)}(t) = 0$ for $s \geq 2$. The IF estimate will be obtained as a solution of [5, 6, 7, 9]:

$$\widehat{\omega}_h(t) = \arg[\max_{\omega \in Q_\omega} \{C_x(t, \omega; \varphi_h)\}] \tag{5}$$

where $Q_\omega = \{\omega : 0 \leq |\omega| < \pi/2T\}$ is a basic frequency interval. The estimation error, at the time-instant t, is [5, 6, 7]:

$$\triangle\widehat{\omega}_h(t) = \omega(t) - \widehat{\omega}_h(t). \tag{6}$$

3. ANALYSIS OF THE ESTIMATION ERROR

Since the IF estimate $\widehat{\omega}_h(t)$ is defined by the stationary point of $C_x(t, \omega; \varphi_h)$, its value follows from $\partial C_x(t, \omega; \varphi_h)/\partial\omega = 0$. The linearization of $\partial C_x(t, \omega; \varphi_h)/\partial\omega = 0$ with respect to: 1) The small estimation error, $\triangle\widehat{\omega}_h(t)$; 2) The residual of the phase deviation, $\triangle\phi$; and 3) Noise ϵ, gives:

$$\frac{\partial C_x(t, \omega; \varphi_h)}{\partial\omega}|_0 + \frac{\partial^2 C_x(t, \omega; \varphi_h)}{\partial\omega^2}|_0 \triangle\widehat{\omega}_h(t) + $$
$$+ \frac{\partial C_x(t, \omega; \varphi_h)}{\partial\omega}|_0 \triangle\delta_{\triangle\phi} + \frac{\partial C_x(t, \omega; \varphi_h)}{\partial\omega}|_0 \delta_\epsilon = 0 \tag{7}$$

where $|_0$ means that the above derivatives are calculated at the point $\omega = \phi'(t)$, $\epsilon = 0$, and $\triangle\phi(t, mT, nT) = 0$. The last two terms in (7) determine the variations of $\partial C_x(t, \omega; \varphi_h)/\partial\omega$ caused by $\triangle\phi$, and ϵ, respectively. The terms from (7) are:

$$\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}|_0 = |A|^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \varphi_h(mT, nT) \times$$
$$\times(-j2nT)e^{j2\phi^{(2)}(t)(mT)(nT)} = 0$$
$$\frac{\partial^2 C_x(t,\omega;\varphi_h)}{\partial\omega^2}|_0 = -|A|^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \varphi_h(mT, nT) \times$$
$$\times(2nT)^2 e^{j2\phi^{(2)}(t)(mT)(nT)} = -4|A|^2 R_h(t)$$
$$\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}|_0 \delta_{\triangle\phi} \cong |A|^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \varphi_h(mT, nT) \times$$
$$\times\triangle\phi(t, mT, nT)(2nT)e^{j2\phi^{(2)}(t)(mT)(nT)} = 2|A|^2 P_h(t). \tag{8}$$

The term $\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}|_0\delta_\epsilon$ will be considered separately. Note that $\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}|_0 = 0$ follows from the kernel $\varphi_h(mT,\, nT)$ symmetry. Using notation, $Q_h = \frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}|_0\delta_\epsilon$, we have:

$$\triangle\widehat{\omega}_h(t) = \frac{1}{2R_h(t)}(P_h(t) + \frac{Q_h}{2|A|^2}). \tag{9}$$

In order to get the IF estimator variance, the term $\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}|_0\delta_\epsilon$ will be calculated by using the inner-product form of CD, [4]:

$$C_x(t, \omega; \varphi_h) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \widetilde{\varphi}_h(mT, nT) \times$$
$$\times[x(t + mT)e^{-j\omega mT}][x(t + nT)e^{-j\omega nT}]^* \tag{10}$$

where $\widetilde{\varphi}_h(mT, nT) = \varphi_h((m+n)T/2, (m-n)T/2)$. Consequently,

$$\frac{\partial C_x(t, \omega; \varphi_h)}{\partial\omega}|_0\delta_\epsilon = j \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \widetilde{\varphi}_h(mT, nT)(n - m)T \times$$
$$[f(t + mT)\epsilon^*(t + nT) + f^*(t + nT)\epsilon(t + mT)]e^{-j\omega(m-n)T}|_0. \tag{11}$$

For the white noise $\epsilon(nT)$, $E\{Q_h\} = 0$. Thus, the estimation variance is:

$$\text{var}\{\triangle\widehat{\omega}_h(t)\} = \frac{\text{var}\{Q_h\}}{16|A|^4 |R_h(t)|^2}, \tag{12}$$

where $R_h(t)$ is defined in (8). By expanding function $\exp(j\, 2\phi^{(2)}(t)(mT)(nT))$ into a power series, $\exp(x) = \sum_{i=0}^{\infty} x^i/i!$, we may represent $R_h(t)$ as:

$$R_h(t) \to h^2 \sum_{i=0}^{\infty} \frac{(j2\phi^{(2)}(t))^{2i}}{(2i)!} h^{4i} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \varphi(t, \tau)t^{2i}\tau^{2i+2} dt d\tau \tag{13}$$

when $h \to 0$, $T \to 0$, $h/T \to \infty$.

Proposition: *Let $\widehat{\omega}_h(t)$ be a solution of (5). For small estimation errors and an FM signal $f(t)$ corrupted by the stationary, white, Gaussian noise, the IF estimators' variance is*

$$\text{var}\{\triangle\widehat{\omega}_h(t)\} = \frac{\sigma_\epsilon^2 C_\varsigma(0, 0; -\left\|\widetilde{\Psi}_h\right\|)}{8|A|^4 |R_h(t)|^2} \tag{14}$$

where $C_\varsigma(0, 0, \left\|\widetilde{\Phi}_h\right\|)$ is a quadratic distribution (with the new kernel $\left\|\widetilde{\Phi}_h\right\| = -\left\|\widetilde{\Psi}_h\right\|$) of the signal $\varsigma(t) = f(t)\exp[-j(\phi'(0)t + \phi(0))]$ at the origin of time-frequency (TF) *plane, and $\left\|\widetilde{\Psi}_h\right\| = \|A_{n-m}\| .* \|\widetilde{\varphi}_h\|$. The $\|\widetilde{\varphi}_h\|$ is a matrix with elements $\widetilde{\varphi}_h(mT, nT)$, while $\|A_{n-m}\|$ is a matrix with elements $A(m,\, n) = n - m$, for $m, n = 1, 2, ..., N$ (N represents assumed finite limits for* m and n). *Operator .∗ denotes element-by-element matrix multiplication.*

Proof:

For real and symmetric kernel $\varphi_h(mT, nT)$ variance may be presented as:

$$\text{var}\{Q_h\} =$$
$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \widetilde{\varphi}_h(m_1T, n_1T)\widetilde{\varphi}_h^*(m_2T, n_2T) \cdot$$
$$\times(n_1 - m_1)T(n_2 - m_2)Te^{-j\omega(m_1-m_2)T}e^{-j\omega(n_2-n_1)T} \times$$
$$\times[f(t + m_1T)f^*(t + m_2T)R^*_{\epsilon\epsilon}(t + n_1T, t + n_2T) +$$
$$+f^*(t + n_1T)f(t + n_2T)R_{\epsilon\epsilon}(t + m_1T, t + m_2T)]|_0. \tag{15}$$

Applying $\widetilde{\varphi}_h(m_1T, nT) = \widetilde{\varphi}_h(nT, m_1T)$ and $R_{\epsilon\epsilon}(t+mT, t+nT) = \sigma_\epsilon^2\delta(m - n)$, we get:

$$\text{var}\{Q_h\} = 2 \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \widetilde{\Phi}_h(m_1T, m_2T) \times$$
$$\times[\varsigma(m_1T)][\varsigma(m_2T)]^* = 2C_\varsigma(0, 0, \Phi_h), \tag{16}$$

where $C_\varsigma(0, 0, \Phi_h)$ is a quadratic TFD (with the new kernel $\widetilde{\Phi}_h(m_1T, m_2T) = \Phi_h((m_1+m_2)T/2, (m_1-m_2)T/2))$ at the origin of TF plane of the signal $\varsigma(t)$. Note that for a linear

FM signal $f(t) = A\exp(jat^2/2)$, we have $\varsigma(t) = f(t)$. For the assumed noise,

$$\widetilde{\Phi}_h(m_1T, m_2T) = \sigma_\epsilon^2 \sum_{n=-\infty}^{\infty} \widetilde{\varphi}_h(m_1T, nT)\widetilde{\varphi}_h^*(m_2T, nT)\times$$

$$\times(n-m_1)(n-m_2)T^2. \quad (17)$$

For finite summation limits this is a matrix multiplication form,

$$\left\|\widetilde{\Phi}_h\right\| = \sigma_\epsilon^2[\|A_{n-m}\| .* \|\widetilde{\varphi}_h\|] \times [\|A_{m-n}\| .* \|\widetilde{\varphi}_h\|] \quad (18)$$

where $\|A_{n-m}\|$ is a matrix with elements $A(m,n) = n - m$, for $m,n = 1,2,...,N$. Elements of matrix $\|\widetilde{\varphi}_h\|$ are $\widetilde{\varphi}_h(mT, nT)$. Let us introduce $\left\|\widetilde{\Psi}_h\right\| = \|A_{n-m}\| .* \|\widetilde{\varphi}_h\|$. Because of symmetry and realness of the kernel $\widetilde{\varphi}_h(mT, nT)$, $\widetilde{\varphi}_h^*(m_2T, nT) = \widetilde{\varphi}_h(nT, m_2T)$, and the asymmetry of matrix $\|A_{n-m}\|$, $\|A_{n-m}\| = -\|A_{m-n}\|$, we have

$$\left\|\widetilde{\Phi}_h\right\| = -\sigma_\epsilon^2 \left\|\widetilde{\Psi}_h\right\|^2. \quad (19)$$

Thus,

$$\mathrm{var}\{Q_h\} = 2\sigma_\epsilon^2 C_\varsigma(0,0; -\left\|\widetilde{\Psi}_h\right\|^2). \quad (20)$$

Substitution of eq.(20) into (12) proves the Proposition. ■

4. SPECIAL CASES OF QUADRATIC DISTRIBUTIONS

The IF estimation variances for the most important and frequently used TFDs follow as special cases from (14).

1. **Pseudo Wigner distribution**: For this distribution $\widetilde{\varphi}_h(mT, nT) = w_h(mT)\delta(m+n)w_h(nT)$, $R_h(t) = \sum_{n=-\infty}^{\infty} w_{h_1}^2(nT) \to Th \int_{-1/2}^{1/2} w^2(\tau)\tau^2 d\tau$, where $w_h(nT)$ is the real and even window function. Thus, we get

$$\mathrm{var}\{\triangle\widehat{\omega}_h(t)\} = \frac{\sigma_\epsilon^2 WD_{|\varsigma|}(t,0;w_{h_2})}{2|A|^4 |R_h(t)|^2} = \frac{\sigma_\epsilon^2}{2|A|^2} W_w \frac{T}{h^3} \quad (21)$$

where $w_{h_1}(nT) = w_h(nT)(nT)$, $w_{h_2}(nT) = w_h^2(nT)\cdot(nT)$, and $W_w = \int_{-1/2}^{1/2} w^4(\tau)\tau^2 d\tau / \left(\int_{-1/2}^{1/2} w^2(\tau)\tau^2 d\tau\right)^2$ is the window $w(\tau)$ dependent constant. Its values for rectangular, Hanning, Hamming and triangular windows are 12, 54.4631, 41.6581 and 34.2857, respectively. Note that for the rectangular window $w_h(nT)$ and stationary, white, Gaussian noise, we get the well known expression from [6]. It can be easily concluded that $\mathrm{var}\{\triangle\widehat{\omega}_h(t)\}$ is not dependent on $\phi^{(2)}(t)$ in the case of linear FM signal.

2. **Spectrogram**: In this case $\widetilde{\varphi}_h(mT, nT) = w_h(mT)$ $w_h(nT)$. Thus

$$\mathrm{var}\{Q_h\} = 2\sigma_\epsilon^2\{ThM_2^{(w^2)}SPEC_\varsigma(0,0;w_h) + \\ +\tfrac{T}{h}M_0^{(w^2)}SPEC_\varsigma(0,0;w_{h_1})\} \quad (22)$$

where $M_r^{(w^2)} = \int_{-1/2}^{1/2} w^2(\tau)\tau^r d\tau$ is the r-th moment of the squared window $w^2(\tau)$. The $R_h(t)$, eq.(13), is

$$R_h(t) = \frac{1}{2|A|^2}\{\mathrm{Re}[STFT_\varsigma(0,0;w_{h_3})STFT_\varsigma^*(0,0;w_h)] -$$

$$-SPEC_\varsigma(0,0;w_{h_1})\} \quad (23)$$

where $w_{h_3}(nT) = w_h(nT)(nT)^2$. Substitution of eqs.(22)-(23) into (12) gives the IF estimator variance in the case of SPEC for any FM signal. In the above equations, $STFT(t,\omega;w_h)$ represents the short-time Fourier transform, while $SPEC(t,\omega;w_h) = |STFT(t,\omega;w_h)|^2$. For the linear FM signal, $f(t) = A\ \exp(jat^2/2)$, we have $SPEC_f(0,0;w_{h_1}) = 0$. Thus,

$$\mathrm{var}\{\triangle\widehat{\omega}_h(t)\} = \frac{\sigma_\epsilon^2 Th}{2} M_2^{(w^2)} \times$$

$$\times \frac{SPEC_f(0,0;w_h)}{\mathrm{Re}^2[STFT_f(0,0;w_{h_3})STFT_f^*(0,0;w_h)]} \quad (24)$$

where, for example, $STFT_f(0,0;w_h) = A\sum_{n=-\infty}^{\infty} w_h(nT)\exp(ja(nT)^2/2)$. Since parameter $\phi^{(2)}(t) = a$ occurs in the exponent of all terms from (24), we can approximate this expression by the exponential form $\sigma_\epsilon^2\exp(P(a))$, where $P(a)$ is a polynomial in a. Expanding $\ln(\mathrm{var}\{\triangle\widehat{\omega}_h(t)\})$ into the Taylor series, the coefficients of the polynomial $P(a)$ may be obtained. This polynomial contains only even powers of a. For relatively small a ($C_w a^2 h^4 < 5$) we may take first two terms of $P(a)$, resulting in very simple approximative expression for the variance

$$\mathrm{var}\{\triangle\widehat{\omega}_h(t)\} \cong \frac{\sigma_\epsilon^2}{2|A|^2}\frac{T}{h^3}S_w e^{a^2 C_w h^4}, \quad (25)$$

where $S_w = M_2^{(w^2)}/(M_2^w)^2$ and

$$C_w = \frac{1}{4}\left((\frac{M_2^w}{M_0^w})^2 + \frac{M_6^w}{M_2^w} - 2\frac{M_4^w}{M_0^w}\right) \quad (26)$$

are the window $w(\tau)$ dependent constants. The M_r^w represents the r-th moment of the window $w(\tau)$. Relative error made by approximation (25) is $1-\exp(O(a^4h^8))$, where $O(\cdot)$ is the Landau symbol. The approximative formula for large values of a may be obtained by applying stationary phase method, [8, 10]. Values of S_w for rectangular, Hanning, Hamming and triangular windows are 12, 28.1135, 19.7324 and 19.2, respectively.

Due to the kernel $\widetilde{\varphi}_h(mT, nT)$ symmetry the same values of variance hold for negative a with $a \to |a|$. Note that in the case of SPEC the IF estimation variance is highly signal dependent. Namely, as a increases, $\mathrm{var}\{\triangle\widehat{\omega}_h(t)\}$ exponentially increases (25) from the value

$$\mathrm{var}\{\triangle\widehat{\omega}_h(t)\} \cong \frac{\sigma_\epsilon^2}{2|A|^2}S_w\frac{T}{h^3}\ ,\ \text{for } a \to 0 \quad (27)$$

that has been derived in the literature [5]. Of course, it holds only for $a \to 0$, while for other values of a the more general relations (24)-(25) derived in this paper hold.

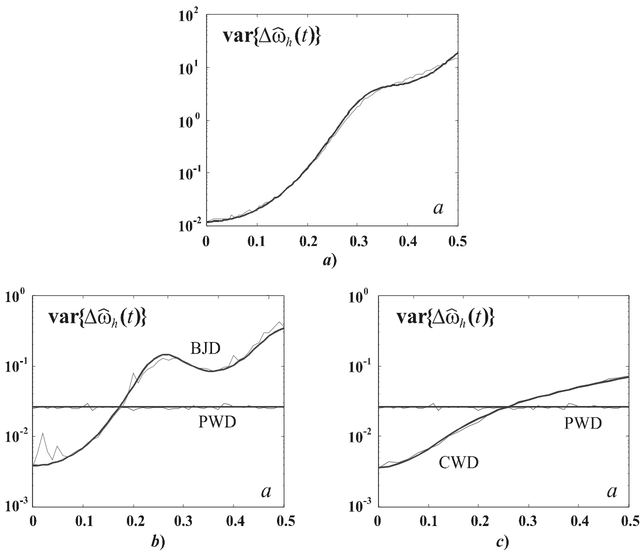

Figure 1: The IF variance obtained theoretically (thick line) and statistically (thin line) for different values of $\phi^{(2)}(t) = a$ for *a*) SPEC, *b*) BJD and PWD, *c*) CWD and PWD.

5. NUMERICAL IMPLEMENTATION

The derived expressions for variance are checked statistically and presented in Figs.1*a*)-*c*). The following quadratic TFDs are considered: *pseudo WD* (PWD), with the Hanning window $w(\tau)$; *spectrogram*; *Born-Jordan* (BJD), $\varphi(mT, nT) = \frac{1}{2|nT|+1} \operatorname{rect}\left|\frac{mT}{2nT}\right|$; and *Choi-Williams distribution* (CWD), $\varphi(mT, nT) = \frac{\sigma}{2\sqrt{\pi}} \cdot \frac{1}{|nT|+1} \exp[-(\frac{\sigma \cdot mT}{2nT})^2]$, $\sigma = \sqrt{2}\pi$. In order to compare these TFDs their parameters are chosen according to the results from [10]. The general expression (14) is used in the numerical analysis. Linear FM signal $f(t) = e^{-j16\pi a t^2}$ corrupted by the stationary white noise with variance $\sigma_\epsilon = 0.25$ is considered. The values of $\phi^{(2)}(t) = a$ with $a \in [0, 1]$ are taken in the case of spectrogram, while $a \in [0, 0.5]$ in the case of other TFDs, when the oversampling is necessary. The signal is considered within time interval $t \in [-2, 2)$, with the sampling period $T = 1/64$. Symmetric kernels, $-h/2 \leq (mT), (nT) \leq h/2$, with $h = 1$ (i.e. 64 samples kernel width) are used.

A very high agreement of theoretical and statistical data may be easily noted from these figures. Statistical data are obtained by running 128 simulations. Note that $\operatorname{var}\{\triangle\widehat{\omega}_h(t)\}$ in the BJD and CWD cases increases (as in the case of the SPEC), as a increases. For small $a \to 0$ they have lower variance than the PWD, while by increasing a they perform worse than the PWD. These conclusions are expected since the RID distributions significantly reduce noise energy located far from the θ, τ axes. For the signals whose ambiguity function lies along the θ, τ axes (as in the case of linear FM signals with $a \to 0$) the RID distributions do not degrade signal representation. On the other hand, for a linear FM signals with larger values of a, RID distributions significantly degrade representation of the analyzed signal. Consequently, in this case it may happen that the TFDs from RID class have worse performance than the WD. A decrease in variance for the BJD, for a between 0.3 and 0.4, is due to its pseudo form.

6. CONCLUSION

In this paper we have performed analysis of the IF estimation based on the general quadratic shift-covariant class of TFD's. The exact variance expressions are derived. The obtained results are checked numerically and statistically.

7. ACKNOWLEDGMENT

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