PRE-STEERING DERIVATIVE CONSTRAINTS FOR ROBUST BROADBAND ANTENNA ARRAYS

Zhang Shutao and Ian Li-Jin Thng
Department of Electrical and Computer Engineering,
National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260, Singapore
Email: eletlj@nus.edu.sg

ABSTRACT

The weights of an optimum PB (Pre-steered broadband) antenna array processor are often obtained by solving a LCMV (linearly constrained minimum variance) problem. The objective function is the mean output power (variance) and the constraints space is a set of linear equations which ensure a constant gain in a fixed direction known as the look direction. However, errors in a practical scenario could degrade the performance of the LCMV processor significantly, namely, mismatches between the look direction and the actual DOA (direction of arrival) of the desired signal, positional errors in the sensors and quantization errors in the pre-steered front end of the broadband processor. The main contribution of this paper is the derivation of a new set of constraints, referred to as the Pre-steering derivative constraints, which is able to maintain the processor robustness in the general 3D (three dimensional) space scenario with all the errors mentioned above.

1. INTRODUCTION

A popular approach for optimizing the weights of the direct form pre-steered broadband antenna array processor [1-2] shown in Fig 1 with $T_1, \ldots T_L$ representing the pre-steering delays is to impose linear equality constraints which fixes the frequency response of the processor in the look direction while minimizing the mean output power [1-3]. This results in the well know *linearly constrained minimum variance* (LCMV) processor. However, the performance of the LCMV processor can be highly degraded when there are imperfections in a practical scenario, namely, directional mismatches between the look direction of the PB processor and the actual direction of arrival of the signal (DOA) of the desired signal, positional errors in sensor locations and quantization errors in the pre-steering delays. Solutions for alleviating directional mismatches in the LCMV include the use of additional *multi-point constraints* [4]. soft quadratic response constraints and maximally flat spatial power response *derivative constraints* [2, 3, 5, 6] in the original LCMV problem. However, the literature has revealed very few techniques to ensure processor robustness in the face of **multiple** imperfections namely, directional mismatch, sensor positional errors and pre-steering delay quantization errors. Thus a new set of constraints referred to as the *pre-steering derivative* constraints for use in the general 3D (three dimensional) space scenario is derived to make the processor robust in the face of multiple errors. The paper is organised as follows: we present the basic notations and equations first, then we revisit look direction constraints and the NS1 (Necessary and Sufficient, First order, Conventional derivative) constraints. After that, we formulate and derive the NS1-PS (Necessary and Sufficient, First order, Pre-steering derivative) constraints for combating a multitude errors in the general 3D-space scenario. Finally we compare the performance of the PB processor subject to the above mentioned three types of constraints under scenarios including directional mismatches, positional errors in the sensors and quantization errors in the pre-steering delays.

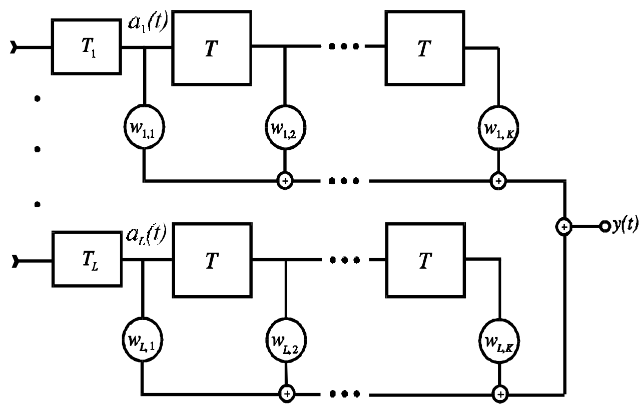

Figure 1: An *L* element, *K* taps direct form pre-steered broadband processor

2. NOTATION

L: Number of sensors; *K:* number of taps (chosen to be an odd number); θ_1 : elevation angle in the 3D plane; θ_2 : azimuth angle; $\otimes$: Kronecker product; *R:* field of real numbers; *Z:* set of integer; *C:* field of complex numbers; Z_n^m : $\{x \in Z, : m \le x \le n\}$; $I_{n\times n}$: $n \times n$ identity matrix; $R^{n\times k}$: $n \times k$ real element matrices; $\boldsymbol{C}^{n\times k}$: $n \times k$ complex element matrices; $\boldsymbol{1}_L$: L-vector with all elements equal to 1; $\boldsymbol{J}_n$: $K \times K$ null matrix except for a diagonal of 1's located on the *n*th sub-diagonal; In the following, (θ_1, θ_2) refers to the DOA of the signal, and $(\theta_{1,PB}, \theta_{2,PB})$ refers to the look direction of the antenna array. It is assumed that the wavefront impinging on the array are plane waves as illustrated in Figure 2. The *l*th sensor position is defined by coordinate (x_l, y_l, z_l) . The propagation delay $\tau_l(\theta_1, \theta_2)$ is evaluated with respect to the coordinate system origin as follows:

$$\tau_l(\theta_1,\theta_2) = (x_l \cos\theta_2 \sin\theta_1 + y_l \sin\theta_2 \sin\theta_1 + z_l \cos\theta_1)/v \in R \quad (1)$$

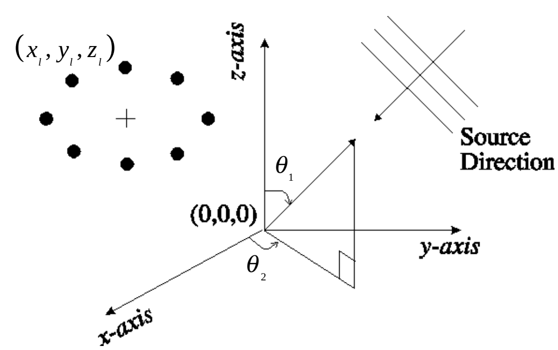

Figure 2: Arbitrary Array Structure in 3D Coordinate System

The state vector of the antenna array is given by:

$$\left[\boldsymbol{a}(t)\right]_{(k-1)L+l} = a_l\left(t + T - kT\right) \in R \ , \ \ k \in Z_K^1 \ \ , \ \ l \in Z_L^1 \qquad (1)$$

The mean output power of the processor is given by

$$p(\boldsymbol{w}) = \boldsymbol{w}^T \boldsymbol{R} \boldsymbol{w} \qquad (2)$$

where $\boldsymbol{R} = \mathrm{E}\left[\boldsymbol{a}(t)\boldsymbol{a}^T(t)\right] \in R^{KL\times KL}$ is the input signal correlation matrix and is positive semi-definite. And $\boldsymbol{w}$ is the stacked vector of real weights of the antenna array processor,

$$\left[\boldsymbol{w}\right]_{(k-1)L+l} = w_{l,k} \in R, \ \ k \in Z_K^1, \ \ l \in Z_L^1 \qquad (3)$$

3. LINEAR CONSTRAINTS AND NS1 CONSTRAINED PB PROCESSOR

The frequency response of the PB processor steered in the look direction $\left(\theta_{1,PB}, \theta_{2,PB}\right)$ can be easily evaluated to be as follows:

$$H(f) = \boldsymbol{w}^T\left[\boldsymbol{d}(f) \otimes \boldsymbol{D}(f)\right]\boldsymbol{s}(f) \qquad (4)$$

where

- $\left[\boldsymbol{d}(f)\right]_k = e^{-j2\pi f\,(k-1)T} \in C \ , \ k \in Z_K^1$
- $\boldsymbol{D}(f) = \mathrm{diag}(e^{-j2\pi f T_l\left(\theta_{1,PB},\theta_{2,PB}\right)}) \in C^{L\times L} \ , \ l \in Z_L^1$, is the complex diagonal matrix of pre-steering delays:
  $T_l(\theta_{1,PB}, \theta_{2,PB}) = T_0 +$
  $$\frac{x_l \cos\theta_{2,PB} \sin\theta_{1,PB} + y_l \sin\theta_{2,PB} \sin\theta_{1,PB} + z_l \cos\theta_{1,PB}}{v} \in R$$
  for T_0 a bulk delay parameter ensuring that,
  $T_l\left(\theta_{1,PB}, \theta_{2,PB}\right) \geq 0 \ , \ l \in Z_L^1$, and
- $\left[\boldsymbol{s}(f, \theta_1, \theta_2)\right]_l = e^{j2\pi f \tau_l\left(\theta_1,\theta_2\right)} \in C \ , \ l \in Z_L^1$

The weights of the look-direction-only PB processor is determined by the following constrained minimization problem [1]:

$$\min_{\boldsymbol{w}} \ p(\boldsymbol{w}) = \min_{\boldsymbol{w}} \ \boldsymbol{w}^T \boldsymbol{R} \boldsymbol{w} \qquad (5)$$

subject to $\qquad \boldsymbol{C}_0 \boldsymbol{w} = \boldsymbol{h}\,. \qquad (6)$

where $\boldsymbol{C}_0 = \boldsymbol{I}_K \otimes \boldsymbol{1}_L^T$ and for the flat frequency response processor [5-6], $\boldsymbol{h}$ is defined follows:

$$\left[\boldsymbol{h}\right]_{k_0} = 1 \ , \ k_0 = (K+1)/2 \text{ , and } \left[\boldsymbol{h}\right]_k = 1 \ , \ k \in Z_K^1, k \neq k_0 \qquad (7)$$

3.1 The NS1 constrained PB Processor

The power response $\rho(f)$ of the PB processor is simply:

$$\rho(f) = H(f) H^*(f) \in R \qquad (8)$$

Relying on the mathematical formulation provided in [6], the NS1 set of constraints include the look direction constraints in (7) plus the following necessary and sufficient first order maximally flat spatial power response constraints:

$$\forall f \in R: \left.\frac{\partial\rho(f)}{\partial\theta_i}\right|_{\substack{\theta_1=\theta_{1,PB}\\ \theta_2=\theta_{2,PB}}} = 0 \Leftrightarrow \left\{\boldsymbol{h}^T\left[\left(\boldsymbol{J}_k - \boldsymbol{J}_{-k}\right) \otimes \boldsymbol{1}_L^T \left.\frac{\partial\boldsymbol{\Lambda}}{\partial\theta_i}\right|_{\substack{\theta_1=\theta_{1,PB}\\ \theta_2=\theta_{2,PB}}}\right]\right\}\boldsymbol{w} = 0$$

$$i \in Z_2^1 \ , \ k \in Z_{K-1}^1 \qquad (9)$$

where $\boldsymbol{\Lambda} = \mathrm{diag}\left[\tau_l\left(\theta_1, \theta_2\right)\right] \in R^{L\times L} \ , \ \ l \in Z_L^1$.

4. THE NS1-PS CONSTRAINED PB PROCESSOR

By observation, the three types of errors will eventually amount to temporal error quantities in the pre-steering delays. Directional mismatches occur when the pre-steer front end is given imprecise information on the DOA of the desired signal. Quantization errors occur when pre-steering delay components are made of discrete components of certain quantisation levels. For positional errors, we refer to a two dimensional scenario in Figure 3. In Figure 3, the desired location of the sensor element is at (x_l, y_l), which is at the center of an imaginary circle called *displacement bound circle* with radius *d*. The radius *d* is referred to as the *displacement bound* of positional errors. However, the actual location of the sensor position is the point (x_l', y_l') within the circle. When the sensor element is steered to the azimuth look direction θ_2, its pre-steering delay is designed to be:

$$T(\theta_2) = T_0 + (x_l \cos\theta_2 + y_l \sin\theta_2)\,/\,v\,. \qquad (10)$$

But in fact should be:

$$T'(\theta_2) = T_0 + (x_l' \cos\theta_2 + y_l' \sin\theta_2)\,/\,v\,. \qquad (11)$$

Hence an error term ε is incurred in the pre-steering delays as follows:

$$\varepsilon = \left|T(\theta_2) - T'(\theta_2)\right| < d\,/\,v \qquad (12)$$

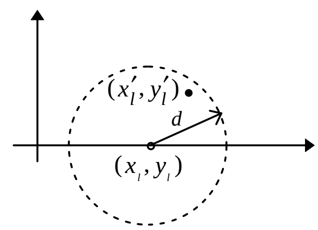

Figure 3: Positional errors in sensor elements

4.2 Derivation of The NS1-PS Constraints

The definition of Nth order NS-PS constraints is as follows:

In a generic 3D-space scenario with directional, positional and quantization errors, the spatial power response of the PB processor in the pre-steered direction $T_{l,0}$, $l \in Z_L^1$ will remain maximally flat of order *N* if and only if all partial derivatives of $\rho(f)$ with respect to T_l , $l \in Z_L^1$ and evaluated at $T_l = T_{l,0}$, $l \in Z_L^1$ up to the *N*th order are zero, i.e.,

$$\forall f \in R: \left.\frac{\partial^n \rho(f)}{\partial T_i^m \partial T_j^{n-m}}\right|_{T_l = T_{l,0}} = 0,$$

$$n \in Z_N^1 \ , \ \ m \in Z_n^1 \cup \{0\} \ \ i, j, l \in Z_L^1 \qquad (13)$$

The last definition follows from the Taylor Expansion of $\rho(f)$ about the designed pre-steering delay set $T_{l,0}$, $l \in Z_L^1$:

$$\rho(f) = \sum_{i=0}^{N} \frac{1}{i!}\left[\sum_{l=1}^{L} \Delta T_l \left.\frac{\partial^i \rho(f)}{\partial T_l}\right|_{T_l = T_{l,0}}\right] \ + \ \upsilon_R \ , \ l \in Z_L^1 \qquad (14)$$

where υ_R represents the remainder terms of the Taylor series expansion. For the case of the NS1-PS set, only the first order derivatives are of concern, namely:

$$\forall f \in R: \left.\frac{\partial\rho(f)}{\partial T}\right|_{T_l = T_{l,0}} = 0 \ , \ \ l \in Z_L^1 \qquad (15)$$

From (16),

$$\frac{\partial \rho}{\partial T_l} = \frac{\partial HH^*}{\partial T_l} = H\frac{\partial H^*}{\partial T_l} + H^*\frac{\partial H}{\partial T_l} = 2\,\mathrm{Re}\left[\frac{\partial H}{\partial T_l}H^*\right] \quad (16)$$

Using (5), $$\frac{\partial H}{\partial T_l} = -j2\pi f\, \boldsymbol{w}^T\left[\boldsymbol{d}(f)\otimes \boldsymbol{\Omega}_l \boldsymbol{D}(f)\right]\boldsymbol{s}(f) \quad (17)$$

where $\boldsymbol{\Omega}_l \in R^{L\times L}$ is a matrix with all zeroes except for a single unity-valued element located at the lth row and lth column.With Kronecker identity $\left[\boldsymbol{P}\otimes\boldsymbol{Q}\right]^H = \boldsymbol{P}^H \otimes \boldsymbol{Q}^H$, we have

$$2\,\mathrm{Re}\left[\frac{\partial H}{\partial T_l}H^*\right] = -4\pi f\ \mathrm{Re}\left\{\boldsymbol{w}^T\left[\boldsymbol{d}(f)\otimes\boldsymbol{\Omega}_l\boldsymbol{D}(f)\right]\boldsymbol{s}(f)\times \boldsymbol{s}^H(f)\left[\boldsymbol{d}^H(f)\otimes\boldsymbol{D}^H(f)\right]\boldsymbol{w}\right\} \quad (18)$$

Using $\boldsymbol{EF}\otimes\boldsymbol{GH} = (\boldsymbol{E}\otimes\boldsymbol{G})(\boldsymbol{F}\otimes\boldsymbol{H})$ and (19), we obtain:

$$2\,\mathrm{Re}\left[\frac{\partial H}{\partial T_l}H^*\right] = -4\pi f\boldsymbol{w}^T\ \mathrm{Re}\{\,j\boldsymbol{d}(f)\boldsymbol{d}^H(f)\otimes \boldsymbol{\Omega}_l\boldsymbol{D}(f)\boldsymbol{s}(f)\boldsymbol{s}^H(f)\boldsymbol{D}^H(f)\}\boldsymbol{w} \quad (19)$$

It can be verified that $\boldsymbol{D}(f)\boldsymbol{s}(f) = e^{-2\pi f T_0}\boldsymbol{1}_L$ and

$$\mathrm{Re}\left[j\boldsymbol{d}(f)\boldsymbol{d}^H(f)\right] = -\sum_{k=1}^{K-1}(\sin 2\pi fkT)(\boldsymbol{J}_k - \boldsymbol{J}_{-k}) \quad (20)$$

substitute (21) into (20), we have:

$$\begin{aligned}\left.\frac{\partial\rho}{\partial T_l}\right|_{T_l=T_{l,0}} &= 4\pi f\sum_{k=1}^{K-1}(\sin 2\pi fkT)\left\{\boldsymbol{w}^T\left[(\boldsymbol{J}_k-\boldsymbol{J}_{-k})\otimes\boldsymbol{\Omega}_l\boldsymbol{1}_L\boldsymbol{1}_L^T\right]\boldsymbol{w}\right\}\\ &= 4\pi f\sum_{k=1}^{K-1}(\sin 2\pi fkT)\left\{\boldsymbol{w}^T\left[(\boldsymbol{J}_k-\boldsymbol{J}_{-k})\otimes\boldsymbol{\Omega}_l\boldsymbol{1}_L\boldsymbol{I}_K\otimes\boldsymbol{1}_L^T\right]\boldsymbol{w}\right\}\\ &= 4\pi f\sum_{k=1}^{K-1}(\sin 2\pi fkT)\left\{\boldsymbol{w}^T(\left[(\boldsymbol{J}_k-\boldsymbol{J}_{-k})\otimes\boldsymbol{\Omega}_l\boldsymbol{1}_L\right]\boldsymbol{h})\right\}\\ &= -4\pi f\sum_{k=1}^{K-1}(\sin 2\pi fkT)\left\{\left[\boldsymbol{h}^T(\boldsymbol{J}_k-\boldsymbol{J}_{-k})\otimes\boldsymbol{1}_L^T\boldsymbol{\Omega}_l\right]\boldsymbol{w}\right\}\end{aligned} \quad (21)$$

and

$$\forall f\in R\ :\ g(f) = \sum_{k=1}^{K-1}a_k\sin(2\pi fkT) = 0 \iff a_k = 0, k\in Z_{K-1}^1 \quad (22)$$

hence, $\forall f\in R$: $\left.\frac{\partial\rho(f)}{\partial T_l}\right|_{T_l=T_{l,0}} = 0 \iff$

$$\left[\boldsymbol{h}^T(\boldsymbol{J}_k-\boldsymbol{J}_{-k})\otimes\boldsymbol{1}_L^T\boldsymbol{\Omega}_l\right]\boldsymbol{w} = 0,\ l\in Z_L^1,\ k\in Z_{K-1}^1 \quad (23)$$

For the case of a flat frequency response PB processor with odd number of taps, then it can be verified that the set of constraints in (23) necessarily and sufficiently reduces to the following set:

$$\left[\boldsymbol{h}^T(\boldsymbol{J}_k-\boldsymbol{J}_{-k})\otimes\boldsymbol{1}_L^T\boldsymbol{\Omega}_l\right]\boldsymbol{w} = 0,\ l\in Z_L^1,\ k\in Z_{(K-1)/2}^1 \quad (24)$$

where vector $\boldsymbol{h}$ satisfies the relationship in (8). In summary, the weights of NS1-PS constrained PB processor is determined by:

$$\min_{\boldsymbol{w}}\ p(\boldsymbol{w}) = \min_{\boldsymbol{w}}\ \boldsymbol{w}^T\boldsymbol{R}\boldsymbol{w} \quad (25)$$

subject to $\boldsymbol{C}_0\boldsymbol{w} = \boldsymbol{h}$. (26)

and $\left[\boldsymbol{h}^T(\boldsymbol{J}_k-\boldsymbol{J}_{-k})\otimes\boldsymbol{1}_L^T\boldsymbol{\Omega}_l\right]\boldsymbol{w} = 0,\ l\in Z_L^1,\ k\in Z_{K-1}^1$ (27)

where (28) can be replaced by (25) for the case of the flat frequency response PB processor.

4.3 Relation between the NS1 and NS1-PS constraints

We now show that the feasibility space of the NS1-PS constraints lies entirely within that of the NS1 constraints. It implies that the NS1-PS constrained PB processor is, at least, as robust as the NS1 constrained PB processor against directional mismatches. Note that the diagonal matrix $\frac{\partial\boldsymbol{\Lambda}}{\partial\theta_i}$ in (10) can be decomposed as follows:

$$\frac{\partial\boldsymbol{\Lambda}}{\partial\theta_i} = \sum_{l=1}^{L}\frac{\partial\tau_l}{\partial\theta_i}\boldsymbol{\Omega}_l \quad (28)$$

Using (28) the constraints in (10) can be decomposed as:

$$\left\{\boldsymbol{h}^T\left[(\boldsymbol{J}_k-\boldsymbol{J}_{-k})\otimes\boldsymbol{1}_L^T\frac{\partial\boldsymbol{\Lambda}}{\partial\theta_i}\right]\right\}\boldsymbol{w} = \sum_{l=1}^{L}\frac{\partial\tau_l}{\partial\theta_i}p_{k,l} \quad (29)$$

where

$$p_{k,l} = \left[\boldsymbol{h}^T(\boldsymbol{J}_k-\boldsymbol{J}_{-k})\otimes\boldsymbol{1}_L^T\boldsymbol{\Omega}_l\right]\boldsymbol{w}\in R,\ l\in Z_L^1,\ k\in Z_{K-1}^1. \quad (30)$$

Clearly,

$$p_{k,l} = 0,\ l\in Z_L^1,\ k\in Z_{K-1}^1 \Rightarrow \left\{\boldsymbol{h}^T\left[(\boldsymbol{J}_k-\boldsymbol{J}_{-k})\otimes\boldsymbol{1}_L^T\frac{\partial\boldsymbol{\Lambda}}{\partial\theta_i}\right]\right\}\boldsymbol{w} = 0. \quad \text{(QED)}$$

5. NUMERICAL STUDIES

Performance of the look-direction-only PB processor, the NS1-constrained and the NS1-PS constrained PB processor in a 2D scenario are presented. For numerical study, a single 6dB broadband directional source located at $\theta_2 = 90^\circ$ with wavelengths ranging from $\lambda_L = 0.125$ to $\lambda_u = 0.25$, normalized relative to the Nyquist sampling frequency of the processor, -30dB white noise and 0 dB broadband isotropic noise spanning one octave of bandwidth, i.e. [0.125Hz, 0.25Hz]. The intertap delay is normalized to 1 second, and number of taps is 7. The PB processor is of the flat frequency response type. The linear array used for numerical study is shown in Figure 4.

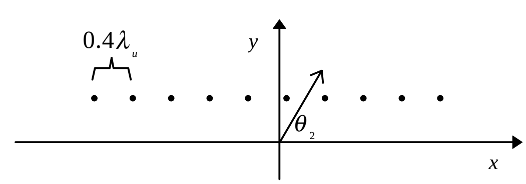

Figure 4: 10 Element Equi-Spaced Linear Array

5.1 Classical Error Scenario

The classical error scenario often used in the literature [1-3, 5, 6] is only directional mismatch with other conditions being ideal. Figure 5 illustrates the resulting spatial power response of the look-direction-only PB processor, the NS1 constrained PB processor, and the NS1-PS constrained PB processor. The spatial power response of the look-direction-only PB processor sports a sharp and slender needle-like tip in the direction of the broadband signal, thus it is not robust in the face of a slight directional mismatch. Both the NS1 and NS1-PS constrained processor have a first order maximally flat tip in the direction of the broadband signal and hence are more robust in the event of a directional mismatch.

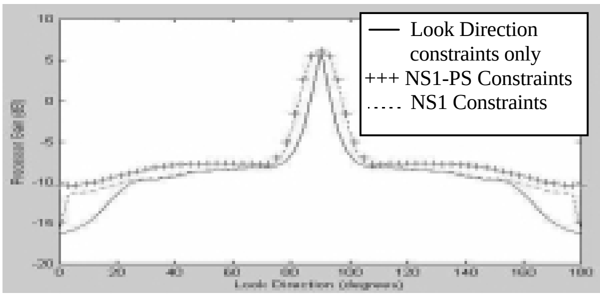

Figure 5: Linear Array (No position and quantization error)

5.2 Positional Error and Quantization Scenario

As illustrated in Figure 3, a random point uniformly generated within the error bound circle serve to be the erred location of the sensor. The displacement bound *d* for the linear array is $0.05\times 0.4\lambda_u$. For quantization error, an 8-bit quantizer is used to represent a total of 256 possible pre-steering delay levels. Figure 6, 7 and 8 illustrate the resulting spatial power response of the above mentioned three types of PB processors in the case of positional, quantizational, and multiple errors.The plots show that the NS1-PS processor remains stoically unaffected by all the error scenarios. However, the error scenario with positional errors and scenario with both positional and quantizational errors lead to a big drop in gain of the look direction only processor

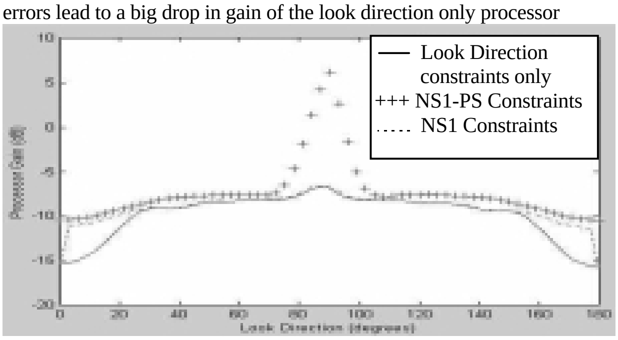

Figure 6: Linear Array with 5% Positional Error

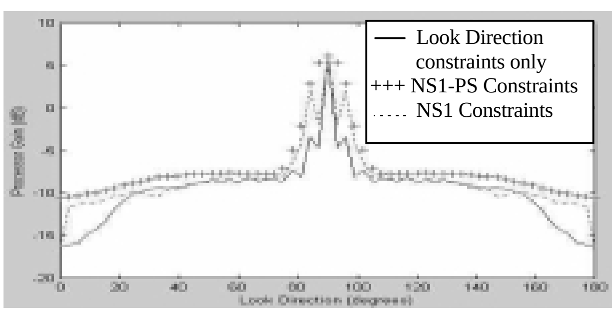

Figure 7: Linear Array with Quantization Error Effects

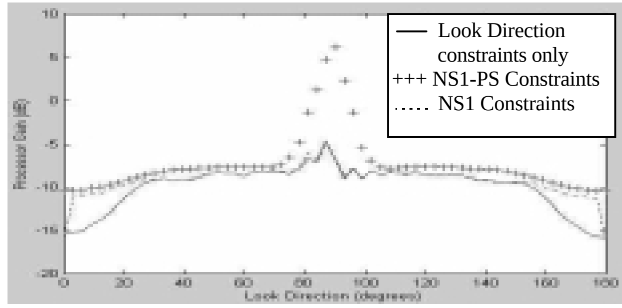

Figure 8: Linear Array with Multiple Errors (Positional and Quantization)

and NS1 constrained processor in the DOA of the signal – larger than 10 dB drop. Quantization error has significantly reduced the width of the spatial power response of the look-direction-only PB processor and the NS1-constrained PB processor in the vicinity of the desired signal. This enhances the sensitivity of the PB processor to directional mismatches which is not desirable as far as robustness is concerned.

6. CONCLUSION

This paper has contributed a new set of constraints for the optimum PB (Pre-steered broadband) array processor to provide robustness against a plethora of errors including directional mismatches, positional errors in the sensor locations and quantization errors in the pre-steering delays. This is a set of necessary and sufficient pre-steering delay derivative constraints which are conveniently linear in characteristic. The new set of constraints also has the property of at least ensuring first order maximally flat spatial power response in the vicinity of the desired signal. Simulation results show that the optimum PB processor whose weights are constrained by the new set of constraints remains stoically robust in the face of multiple errors while the optimum PB processor whose weights are constrained by conventional derivative constraints suffer either gain degradation in the vicinity of the desired signal or result in undesirable needle-like spatial power responses in the vicinity of the desired signal.

REFERENCES

[1] O. L. Frost III, "An Algorithm for Linearly Constrained Adaptive Antenna Array Processing", Proc. IEEE, vol. 60, no. 8, August 1972, pp. 926-935.

[2] K. M. Buckly and L. J. Griffiths, "An Adaptive Generalized Sidelobe Canceller with Derivative Constraints", *IEEE Transactions of Antennas and Propogation,* vol. 34, no.3, Mar 1986, pp. 311-319

[3] J. Tuthill, Y. H. Leung and I. L. Thng, "Adaptive RLS Filters with Linear and Quantization Constraints", *Proc.* ICASSP'95, Detroit, USA, 1st April 1995, pp. 1424-1427.

[4] D. Nunn, "Performance assessments of a time-domain adaptive antenna processor in a broadband environment", *Proc. Inst. Elect. Eng.*, vol. 130, pts. F and H, no. 1, Feb. 1983.

[5] M. H. Er and A. Cantoni, "Derivative constraints for broad-band element space antenna array processors", *IEEE Transactions on Acoustic, Speech and Signal Processing,* vol. 31, Dec 1983, pp. 1378-1393.

[6] I. Thng, A. Cantoni and Y. H. Leung, "Constraints for Maximally Flat Optimum Broadband Antenna Arrays", *IEEE Trans. Signal Processing,* vol. 43, no.6, June 1995, pp. 1334-1347.