WEIGHTED LEAST SQUARES METHOD FOR THE APPROXIMATION OF DIRECTIONAL DERIVATIVES

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ABSTRACT

Using the facet model we design a family of filters for the approximation of partial derivatives of the digital image surface. Prior information (e.g., local dominant orientation) are incorporated in a two dimensional weight function. A weighted least squares estimation of the facet parameters is applied in order to design the proposed filters. Exemplary application of the proposed filters to fingerprint image segmentation is also presented.

1. INTRODUCTION

Facet models have found various applications in computer vision and image processing. This applications include edge detection [1, 2, 3], topographic feature extraction [4], optical flow estimation [5], shape from shading [6].

In essence, the facet model treats the digital image as a noisy observation of an underlying piecewise continuous surface. Using a facet model, various local properties of the digital image (e.g. directional derivatives, topographic features) can be interpreted by means of the corresponding properties of the underlying image surface.

A parametric form of the underlying surface is assumed. Fitting the surface over the intensity values of the image pixels in a certain neighborhood is therefore reduced to the estimation of the surface parameters.

In this paper we use a weighted least squares (WLS) method in order to estimate the parameters of the underlying surface. The proposed scheme allows us to introduce prior information about image by weighting differently the importance of different image pixels in the estimation process. Using this technique we develop a family of filters which may be used in order to approximate the digital image as well as the directional derivatives of the image surface. An application of the proposed filters to fingerprint image segmentation is presented. The image is segmented based on the sign of the largest curvature of the image surface, which is estimated using the proposed filters. The local ridge orientation in different regions of the fingerprint image as well as the average inter-ridge distance constitute the prior information used to design the filters.

The organization of the paper is as follows. The formulas of the proposed filters are derived in the Section 2. Section 3 presents the fingerprint segmentation technique based on the sign of the digital image curvature. Experimental results showing the importance of incorporating prior information in the filter design are shown in Section 4. Finally some conclusions are presented in the last section of the paper.

2. THE PROPOSED FILTERS

In the following discussion the neighborhood of a pixel (i,j) denotes a rectangular region of size $(2N+1)\times(2N+1)$ pixels which has the pixel (i,j) in the center. The problem we consider is to determine a continuous surface which fits over the gray values of the pixels located in the neighborhood of the pixel (i,j). This surface, called also the underlying surface of the image [1], carries our assumptions about the local image structure.

For simplicity we restrict the class of continuous surfaces to the two-dimensional (2D) functions of the form

$$f(x,y)=\sum_{k=0}^{K}\sum_{\ell=0}^{K}c_{k,\ell}h_k(x)h_\ell(y), \tag{1}$$

where $\{c_{k,\ell}\}_{0\le k,\ell\le K}$ are the unknown parameters of the surface, and $\{h_k(x)\}_{0\le k\le K}$ denotes a set of $K+1$ one-dimensional (1D) continuous functions defined on $[-N,N]$. In addition, we impose an orthogonality condition between the discrete versions of the continuous functions h_k. This is

$$< h_k, h_\ell >= \sum_{n=-N}^{N} h_k(n)h_\ell(n)=0, \text{ for any } k\neq l. \tag{2}$$

As an example a proper set of functions $\{h_k(x)\}$ may consists of discrete Chebyshev polynomials constructed as in [1]. First, the zero order polynomial is assumed $h_0(x)=1$, and then higher order polynomials are successively constructed such that to preserve the orthogonality condition (2). This is, the k coefficients of the polynomial $h_k(x)=x^k+a_{k-1}x^{k-1}+\cdots+a_0$ are determined by solving the linear system of k equations $< h_k, h_\ell >=0$, where $\ell=0,\ldots,k-1$. The first four polynomial functions constructed as above are

$$\begin{aligned}
h_0(x) &= 1,\\
h_1(x) &= x,\\
h_2(x) &= x^2-N(N+1)/3,\\
h_3(x) &= x^3-x(3N^2+3N-1)/5.
\end{aligned}$$

The gray values of the pixels located in the neighborhood of (i,j) are considered as noisy observations of the continuous underlying surface $f(x,y)$. The image formation model is thereby expressed as

$$g(i+n, j+m) = f(n,m) + \xi(n,m), \; -N \leq n,m \leq N, \; (3)$$

where $g(i+n, j+m)$ denotes the gray value of an image pixel located in the neighborhood of (i,j), and $\xi(n,m)$ denotes the residual error between the gray value of the image pixel and the gray value of the underlying surface. In matrix notation the equations (3) can be written as

$$\mathbf{G} = \mathbf{H}\mathbf{C}\mathbf{H}^T + \mathbf{\Xi}, \quad (4)$$

where $\mathbf{G}$ is a $(2N+1)\times(2N+1)$ observation matrix containing the gray values of all pixels in the neighborhood of the pixel (i,j)

$$\mathbf{G} = \begin{bmatrix} g(i+N, j+N) & \cdots & g(i+N, j-N) \\ \vdots & \ddots & \vdots \\ g(i-N, j+N) & \cdots & g(i-N, j-N) \end{bmatrix},$$

$\mathbf{H}$ is a $(2N+1)\times(K+1)$ matrix given by

$$\mathbf{H} = \begin{bmatrix} h_0(N) & \cdots & h_K(N) \\ \vdots & \ddots & \vdots \\ h_0(-N) & \cdots & h_K(-N) \end{bmatrix},$$

$\mathbf{\Xi}$ denotes the $(2N+1)\times(2N+1)$ residual error matrix, and $\mathbf{C} = (c_{k,\ell})_{0\leq k,\ell \leq K}$ is the $(K+1)\times(K+1)$ matrix storing the unknown parameters $c_{k,\ell}$.

Using a matrix to vector transformation the equation (4) can be rewritten as

$$\text{vec}(\mathbf{G}) = (\mathbf{H}\otimes\mathbf{H})\text{vec}(\mathbf{C}) + \text{vec}(\mathbf{\Xi}), \quad (5)$$

where vec() is an operator that maps the elements of a matrix into a vector by row ordering, and $\otimes$ denotes the Kronecker product operator.

At this moment, it is straightforward to compute the LS estimator of the parameter matrix $\mathbf{C}$ by minimizing $\text{vec}(\mathbf{\Xi})^T\text{vec}(\mathbf{\Xi})$. Rather than do so, we prefer to introduce some prior information in the estimation of the unknown parameters. We consider that the prior information is modeled in the form of a discrete 2D function $\{w(n,m)\}_{-N\leq n,m\leq N}$ which associates a weight value $w(n,m)$ to each image pixel $(i+n, j+m)$ in the neighborhood of (i,j). The weight of a pixel reflects the importance the pixel has in the estimation of the underlying surface. Examples of weight functions based on prior geometrical information will be presented in the next section in the context of fingerprint image segmentation. To simplify the notations we consider that the weights $\{w(n,m)\}_{-N\leq n,m\leq N}$ are stored in a weight matrix $\mathbf{W}$ of size $(2N+1)\times(2N+1)$.

The WLS fitting of the continuous surface $f(x,y)$ over the digital image in the neighborhood of the pixel (i,j) is performed by determining the parameters $c_{k,\ell}$ such that to minimize the weighted residual error

$$\mathcal{E} = \text{vec}(\mathbf{\Xi})^T\mathcal{W}\text{vec}(\mathbf{\Xi}), \quad (6)$$

where $\mathcal{W} = \text{diag}(\text{vec}(\mathbf{W}))$.

The WLS estimator of the parameter matrix is given by

$$\text{vec}(\widehat{\mathbf{C}}) = (\mathcal{H}^T\mathcal{W}\mathcal{H})^{-1}\mathcal{H}^T\mathcal{W}\text{vec}(\mathbf{G}), \quad (7)$$

where $\mathcal{H} = \mathbf{H}\otimes\mathbf{H}$.

The (p,q) order partial derivative of the discrete image in the pixel (i,j) is approximated by the corresponding partial derivative of the underlying surface in $(0,0)$

$$g^{(p,q)}(i,j) \approx \sum_{k=0}^{K}\sum_{l=0}^{K} \hat{c}_{k,l} h_k^{(p)}(0) h_\ell^{(q)}(0), \quad (8)$$

which, in matrix notation is equivalent with

$$g^{(p,q)}(i,j) \approx \mathbf{h}_p^T\widehat{\mathbf{C}}\mathbf{h}_q = (\mathbf{h}_q\otimes\mathbf{h}_p)^T\text{vec}(\widehat{\mathbf{C}}), \quad (9)$$

where $\mathbf{h}_p = [h_0^{(p)}(0) \; \cdots \; h_K^{(p)}(0)]^T$ stores the p-th order derivatives in 0 of the $K+1$ continuous functions $\{h_k\}_{0\leq k\leq K}$.

Replacing (7) in (9) we obtain the approximation formula for the (p,q) order partial derivative of the image surface in (i,j)

$$g^{(p,q)}(i,j) \approx (\mathbf{h}_q\otimes\mathbf{h}_p)^T(\mathcal{H}^T\mathcal{W}\mathcal{H})^{-1}\mathcal{H}^T\mathcal{W}\text{vec}(\mathbf{G}), \quad (10)$$

which, computed for all pixels (i,j) is equivalent with filtering the image with the $2D$ filter

$$\mathbf{F}_{p,q} = \text{vec}^{-1}\left((\mathbf{h}_q\otimes\mathbf{h}_p)^T(\mathcal{H}^T\mathcal{W}\mathcal{H})^{-1}\mathcal{H}^T\mathcal{W}\right), \quad (11)$$

where $\text{vec}^{-1}()$ is the inverse of vec() operator. The filter (11) has a region of support of $(2N+1)\times(2N+1)$, and in general, it is not a separable filter.

One can show that in the particular case when all the weights are equal the filter is separable. This is, using the properties of the Kronecker product the equation (10) is equivalent with

$$g^{(p,q)}(i,j) \approx (\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{h}_p)^T\mathbf{G}(\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{h}_q), \quad (12)$$

where $\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{h}_p$, and $\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{h}_q$ are the impulse responses of two 1D filters acting respectively along the columns and lines of the digital image.

3. APPLICATION OF THE PROPOSED FILTERS TO FINGERPRINT SEGMENTATION

A fingerprint image exhibits a quasiperiodic pattern of ridges (dark regions) and valleys (light regions). The local ridge characteristics called minutiae form a pattern which is unique for each fingerprint [7]. The fingerprint image segmentation in ridge and valley regions heavily influences the performances of the minutiae extraction process and hence the performances of the overall system of automated fingerprint identification. The segmentation process classifies each image pixel in one of the two aforementioned regions. The binary image obtained follows to be used further by subsequent processes in order to detect and classify the minutiae points $[8-10]$.

Inspecting a fingerprint image, we may note that the neighborhood ridges are almost parallel and their orientation is slowly changing in most of the fingerprint image area. Suppose that we construct a one dimensional sequence by collecting the gray values of the pixels located along a short segment orthogonal to the local ridge orientation. The sequence usually exhibits an almost sinusoidal shape with low and high gray values corresponding to pixels located respectively on the ridges, and on the valleys intersected by our segment. The discrimination between the two classes can be therefore performed based on the sign of the second order derivative of the one dimensional sequence. The positive and negative values of this derivative correspond to pixels located on ridges and valleys respectively. At the image level, we have therefore to investigate the sign of the second order directional derivative of the image surface along the direction orthogonal to ridge orientation. We approximate this direction with the direction of largest curvature of the image surface, and hence the second order directional derivative is given by the largest eigenvalue of the Hessian matrix [11].

The Hessian matrix of the digital image (g) in the pixel (i, j) is approximated by

$$\mathcal{D}(i,j) = \begin{bmatrix} g^{(2,0)}(i,j) & g^{(1,1)}(i,j) \\ g^{(1,1)}(i,j) & g^{(0,2)}(i,j) \end{bmatrix}. \quad (13)$$

Let $\lambda_1(i,j)$, and $\lambda_2(i,j)$ denote the two eigenvalues of the Hessian matrix, with $|\lambda_1(i,j)| \geq |\lambda_2(i,j)|$. The sign of the largest surface curvature ($\lambda_1(i,j)$) in the neighborhood of (i,j) is given by

$$\text{sign}(\lambda_1(i,j)) = \text{sign}(g^{(2,0)}(i,j) + g^{(0,2)}(i,j)). \quad (14)$$

The pixel (i,j) is classified as a ridge pixel if $\lambda_1(i,j) > 0$, and as a valley pixel otherwise. The segmentation problem is thereby reduced to the problem of estimating the second order derivatives of the digital image in each pixel. This estimation may be performed using the filters design in the previous section. Specifically, from (11) and (14) we obtain the formula of a filter which may be used to compute the sign of $\lambda_1(i,j)$ in all image pixels. This is

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_{2,0} + \mathbf{F}_{0,2} & (15)\\ &= \text{vec}^{-1}\left((\mathbf{h}_0 \otimes \mathbf{h}_2 + \mathbf{h}_2 \otimes \mathbf{h}_0)^T (\mathcal{H}^T \mathcal{W} \mathcal{H})^{-1} \mathcal{H}^T \mathcal{W}\right). \end{aligned}$$

4. EXPERIMENTAL RESULTS

In this section we show some examples of fingerprint image segmentation using the proposed filters.

Let us consider first the case when no prior information is used to design the filters, and hence all weight values ($w(n,m)$) are chosen equal. We use Chebyshev polynomials up to the third degree for designing the filters. The derivatives in each image pixel are estimated by taking into account the gray values of the pixels located in a $(2N+1) \times (2N+1)$ neighborhood. Changing the size of this neighborhood we may obtain different segmentation results as shown in Fig. 1.

Fig. 1. Example of fingerprint image segmentation based on the sign of the largest curvature of the underlying surface. From left to right and up to bottom, the first image represents the original image and the following three images represent the segmented versions obtained by approximating the derivatives in each image pixel based on neighborhoods of 5×5, 7×7 and 11×11 respectively.

One may note that the segmentation results are highly affected by the size of the neighborhood. Both very small and very wide neighborhood windows introduce false minutiae. Small neighborhood sizes may result in too accurate approximations, that are not able to reduce the noise present in the image. On the other hand, increasing the size of this neighborhood the underlying surface may become too smooth for the signal, mixing the neighborhood ridges and valleys.

A good practice consists of adapting the neighborhood to local image geometry. This is, in order to enhance the definition of ridges against valleys the underlying surface must accurately approximate the gray values along the direction orthogonal to the local ridge orientation. On the other hand a less accurate approximation would be required along the ridge and valley lines, in order to cancel short ridge breaks, and small irregularities.

Our solution consists of adapting the shape of the neighborhood by weighting differently the importance of each pixel. The weight function use in our experiments is

$$w(n,m) = \exp\left(-\frac{x_\theta^2}{2\sigma_x^2} - \frac{y_\theta^2}{2\sigma_y^2}\right), \; -N \leq n, m \leq N, \quad (16)$$

with

$$\begin{aligned} x_\theta &= n\cos\theta + m\sin\theta, & (17)\\ y_\theta &= m\cos\theta - n\sin\theta, & (18) \end{aligned}$$

where θ is the local ridge orientation, and $\sigma_x \gg \sigma_y$. The value of σ_y is chosen three times smaller than the average inter-ridge distance.

Fig. 2. Original image (left), and the segmented image obtained when prior information is used to design the filters (right).

In our experiments the image is first divided in non-overlapping blocks of 8×8 pixels, and then the ridge orientation inside each block is estimated using an LS estimator [12]. Quantizing the orientation angle on 8 levels between 0 and π we need a number of 8 filters to perform the segmentation. These filters are precomputed using (15) and (16).

The segmentation result obtained with the proposed filters for the same image shown in Fig. 1, is shown in Fig. 2.

A comparison between the segmentation achieved using the proposed filters and two other segmentation techniques is shown in Fig. 3.

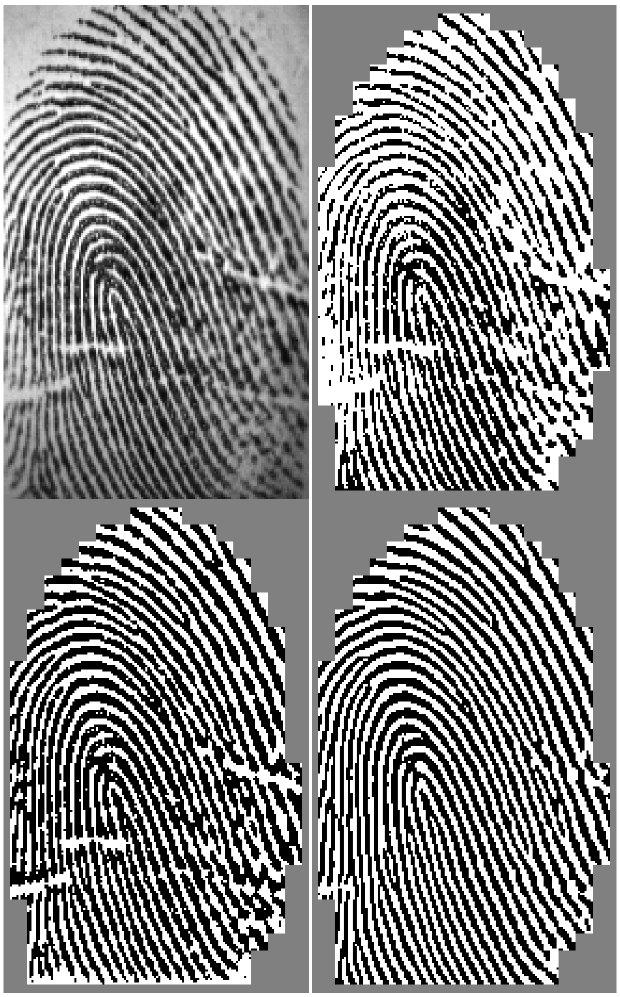

Fig. 3. Fingerprint segmentation example. From left to right and up to bottom we have: the original image, the segmented image obtained with a global threshold, the segmented image when a local threshold is used, and the segmented image obtained with the proposed method.

The image segmented with the proposed filters exhibits less false minutiae than in the other cases. Because of the manner the weight function is chosen, small irregularities along the ridges (e.g., scars) are deleted.

5. CONCLUSIONS

A family of 2D filters for approximation of the partial derivatives of the image surface has been introduced. The filters are design based on the facet model using a weighted least square estimation of the facet parameters. The WLS scheme allows us to design the filters by taking into account prior information about the digital image. An application of the proposed filters to fingerprint image segmentation has been presented. The fingerprint image is segmented based on the sign of the largest principal curvature of the image surface which is approximated using the proposed filters. The importance of including prior information (e.g. local ridge orientation, average inter-ridge distance) in the design of the filters is pointed out by different segmentation examples.

6. REFERENCES

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