BOUNDS ON THE CHANNEL DISTORTION OF VECTOR QUANTIZERS

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ABSTRACT

Vector-Quantization (VQ) is a widely implemented method for low-bit-rate signal coding. A common assumption in the design of VQ systems is that the digital information is transmitted through a perfect channel. Under this assumption, the assignment of channel symbols to the VQ Reconstruction Vectors (RV) is of no importance. However, under physical channels, the effect of channel errors on the VQ system performance depends on the index assignment of the RV. For a VQ of size *N*, there are $N!$ possible assignments, meaning that an exhaustive search over all possible assignments is practically impossible. In this paper, lower and upper bounds on the performance of VQ systems under channel errors, over all possible assignments, are presented. A related expression for the average performance is also discussed. Numerical examples are given in which the bounds and average performance are compared with index assignments obtained by the index-switching algorithm.

1. INTRODUCTION

A typical *Vector Quantization* (VQ) based communication system is shown in Fig. 1.

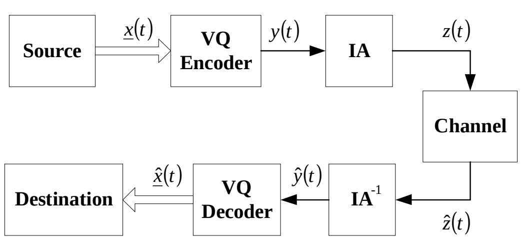

Fig.1 - Vector Quantization based Communication system

A discrete-time *Source* emits signal samples over an infinite (or densely finite) alphabet. The *VQ Encoder* translates source output vectors into *Channel* digital sequences. The *VQ decoder*'s goal is to reconstruct source samples from this digital information and deliver then to the *Destination*. Since the analog information cannot be perfectly represented by the digital information some *Quantization Distortion* must be tolerated.
In each channel transmission the VQ encodes a *K*-dimensional vector of source samples - $\underline{x}(t)$ into a *Reconstruction Vector index* $y(t)$, where the discrete variable *t* represents the time instant or a channel-use counter. The index is taken from a finite alphabet, $y(t)\in\{0,1,\ldots,N-1\}$, where N is the VQ size (number of reconstruction vectors and number of possible channel symbols).
The *Index Assignment* (IA) is represented in Fig. 1 by a permutation operator,

$$\Pi:\quad y(t)\in\{0,1,\ldots,N-1\}\rightarrow z(t)\in\{0,1,\ldots,N-1\} \qquad (1)$$

The number of possible permutations, $N!$, increases very fast with *N*, e.g., for just 4-bits indices there are $16!\approx 2\cdot 10^{13}$ possible permutations. For typical values of N, examination of all possible permutations is therefore impractical.
The channel index $z(t)=\Pi\{y(t)\}$ is sent through the channel. For *Memoryless Channels*, The channel output $\hat{z}(t)$ is a random mapping of its input $z(t)$, characterized by the *Channel Probability Matrix* Q, defined by:

$$\{Q\}_{ij} = \mathrm{Prob}\{\hat{z}(n)=j|z(n)=i\} \qquad (2)$$

Throughout we shall assume that Q is symmetric (i.e., *Symmetric Memoryless Channels*). For the special case of the *Binary-Symmetric-Channel* (BSC):

$$\{Q\}_{ij} = \mathrm{Prob}\{\hat{z}(n)=j|z(n)=i\}=q^{H(i,j)}(1-q)^{L-H(i,j)} \qquad (3)$$

where *L* is the number of bits ($N=2^L$) per channel use, *q* is the *Bit-Error-Rate* (BER) and $H(i,j)$ is the *Hamming Distance* between the binary representations of *i* and *j*.
At the receiver, after inverse-permutation, the *VQ Decoder* converts the channel output symbols into one of *N* possible reconstruction vectors - $\hat{\underline{x}}(t)$. The fidelity of the transmission is defined by a distortion measure between the input and the output of the VQ system $d(\underline{x},\hat{\underline{x}})$.
Knowledge of the source statistics $p(\underline{x})$ or a representing *Training Sequence* is assumed. The performance of the overall system is measured in terms of the average distortion $E[d(\underline{x},\hat{\underline{x}})]$.
In "classic" discussions of VQ applications, the channel is assumed to be noiseless ($Q=I$, where I is the unity matrix), [1], so that no errors occur during transmission and $y(t)=\hat{y}(t)$ for every *t*. This assumption is based upon using a channel encoder-decoder pair to correct channel errors, causing the distortion due to channel-errors to be negligible. The permutation Π has no effect in this case.
Upon knowledge of the source statistics, Lloyd's algorithm [1] may be used to design the VQ. In Practice, a training sequence is used and the LBG algorithm [1] is implemented. Both methods are iterative and alternately apply the *Nearest-Neighbor* Condition and the *Centroid* condition.

In some applications, channel coding is not utilized due to complexity or bit-rate constraints. In that case, if a channel error occurs, a wrong reconstruction vector is selected at the decoder. The distortion due to channel errors can be significant and affects the design of the VQ system [2-9].

In the literature two main approaches are proposed to improve the performance of vector quantizers under channel errors. The first method allows modification of the partition regions and their corresponding codevectors. This modification results in a *Weighted-Nearest-Neighbor* and *Weighted-Centroid* conditions [4],[5]. The second approach is trying to reduce channel distortion by using better index assignments. The search for an optimal index assignment is a special case of the Quadratic-Assignment Problem and is known to be NP-Complete [5].

Several suboptimal methods are suggested in the literature. In [7], [8] an iterative index-switching algorithm is proposed. After selecting an initial assignment, the algorithm searches for a better assignment by exchanging indices of codevectors. It keeps the new assignment if it performs better than its predecessor. This algorithm can only offer a local minimum. A more sophisticated algorithm is examined in [5], where Simulated Annealing is used to search for an optimal index assignment. For the special case of a Uniform Scalar Quantizer and the Uniform Source under the Binary Symmetric Channel, it is shown in [3] that the Natural Binary Code assignment is an optimal assignment.

The difficulty in obtaining good assignments and the need to estimate the performance of a given assignment validates the development of performance bounds and a related expression for the average performance over all possible index assignment. Given a VQ structure, upper and lower bounds on the "Assignment Gain" benefits the VQ designer who is searching for an efficient assignment. The expression for the average performance over all index assignments can help in determining how well a given assignment performs.

The remainder of the paper is organized as follows. In section 2, the distortion due to channel errors is defined. In section 3, bounds on the performance of a given VQ system under a symmetric memoryless channel, over all possible index assignments, are obtained. A related expression for the average performance over all index assignments is also presented in section 3. Numerical results are presented and discussed in section 5, while conclusions are given in section 6.

2. CHANNEL DISTORTION

The Vector Quantization system consists of a partition of the signal space Ω of all possible input vectors - $\underline{x}$. This space is partitioned into *N* regions, R_i , $i=0,1,\ldots,N-1$. These regions cover the whole signal space and are nonoverlapping. Each partition region R_i has a corresponding *Reconstruction (*or *Representation) Vector* - $\underline{\phi}_i$.

The encoder's input is a *K*-dimensional source vector - $\underline{x}$. The symbol $y(t)=i$ is emitted if $\underline{x}\in R_i$. The corresponding channel symbol, $z(t)=\Pi(i)$, is transmitted through the channel. The channel's output is a random mapping of this transmission. Upon receiving the channel symbol $\hat{z}(t)=j$ the decoder emits the reconstruction vector that corresponds to the index $\Pi^{-1}(j)$.

The overall distortion of the VQ-based communication system is:

$$D_T = E\left[d\left(\underline{x},\underline{\hat{x}}\right)\right] = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\left\{\pi\cdot Q\cdot\pi^T\right\}_{ij}\int_{R_i} d\left(\underline{x},\underline{\phi}_j\right)p\left(\underline{x}\right)d\underline{x} \quad (4)$$

In (4) the permutation is represented by a permutation matrix - π , whose entries are 0's and 1's and the sum of each of its rows and columns is 1.

For the perfect channel, $Q=I$, the permutation matrix π is of no importance, and the only factor affecting system performance is the *Quantization Distortion*:

$$D_{Q=I} = E\left[d\left(\underline{x},\underline{\hat{x}}\right)\right]_{Q=I} = \sum_{i=0}^{N-1}\int_{R_i} d\left(\underline{x},\underline{\phi}_i\right)\cdot p\left(\underline{x}\right)\cdot d\underline{x} \quad (5)$$

In the following analysis, another factor, the *Channel Distortion* is defined by:

$$D_C = \sum_{i=0}^{N-1} p_i \sum_{j=0}^{N-1}\left\{\pi Q\pi^T\right\}_{ij}\cdot d\left(\underline{\phi}_i,\underline{\phi}_j\right) = trace\left\{P\pi Q\pi^T D\right\} \quad (6)$$

where p_i is the probability of $\underline{x}\in R_i$:

$$p_i = \int_{R_i} p\left(\underline{x}\right)\cdot d\underline{x} \quad (7)$$

The matrix P in (6) is a diagonal matrix, which contains these probabilities, i.e., $P=diag\left\{p_0,p_1,\ldots,p_{N-1}\right\}$, and the entries of the matrix D are the distances between all possible pairs of reconstruction vectors: $D_{ij}=d\left(\underline{\phi}_i,\underline{\phi}_j\right)$.

It is shown in [4] that for the Euclidean distance measure, and Centroid quantizers the overall distortion is sum of the quantization and channel distortions: $D_T = D_Q + D_C$.

3. AVERAGE PERFORMANCE AND BOUNDS

In this section we introduce lower and upper bounds on the channel distortion, as defined in (6), under Symmetric Memoryless Channels over all possible assignments (or permutation matrices - π). A related expression for the average performance is also obtained. We define a symmetric matrix $\hat{D}$ as $\hat{D}=DP+P^TD^T$. Using the symmetry property of the Channel Matrix, Q , the channel distortion then becomes:

$$D_C = \frac{1}{2}trace\left\{Q\pi^T\hat{D}\pi\right\} \quad (7)$$

A fundamental step in the development of the bounds the replacement of the matrix $\hat{D}$ in (7) by another symmetric matrix $\tilde{D}$, such that it has the all-one vector $\underline{1}=\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ as an eigenvector, while D_C is just changed by a known constant. To this end, we define a "Column Structured" matrix as:

$$C_i = \begin{bmatrix} 0 & & 0 & 1 & 0 & & 0 \\ 0 & & 0 & 1 & 0 & & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & & 0 & 1 & 0 & & 0 \end{bmatrix} \quad (8)$$

$\uparrow i$ – th column

Recalling that Q represents probabilities, the sum of any of its rows is one, so the vector $\underline{1}$ is an eigenvector of Q: $Q \cdot \underline{1} = \underline{1}$. The same argument is valid for the columns of the matrix C_i: $Q \cdot C_i = C_i$, $i = 0,1,\ldots,N-1$.

We now define a symmetric *Cross-Structured Matrix* to be $\alpha(C_i + C_i^T)$, where α is a scalar. It is shown in [2] that, regardless of the permutation matrix π, adding a *Cross Structured* matrix to the matrix $\hat{D}$ changes the expression in (7) just by the addition of the scalar α:

$$\frac{1}{2} trace\left\{ Q\pi^T \left[\hat{D} + \alpha \left(C_i^T + C_i \right) \right] \pi \right\} = \frac{1}{2} trace\left\{ Q\pi^T \hat{D} \pi \right\} + \alpha \qquad (9)$$

Further, let s_i be the sum of the elements in the *i*-th row of the matrix $\hat{D}$:

$$s_i = \sum_{j=0}^{N-1} \hat{D}_{ij} , \qquad (10)$$

and denote by *k* the index of the row having the largest sum: $k = \arg\max_i \{s_i\}$. In order to achieve the desired property $\tilde{D} \cdot \underline{1} = \omega_0 \underline{1}$, for some ω_0, all rows of $\tilde{D}$ must have the same sum of entries. Let us examine the effect of adding a "*Cross Structured*" matrix $\alpha(C_i + C_i^T)$ to a general matrix M of size $N \times N$. The sum of elements in all rows except for the *i*-th row is increased by α, while the sum of elements in the *i*-th row is increased by $(N+1) \cdot \alpha$.

The matrix $\tilde{D}$ is therefore

$$\tilde{D} = \hat{D} + \sum_{i=0}^{N-1} \alpha_i \left(C_i + C_i^T \right) \qquad (11)$$

where

$$\alpha_i = 1/N \left(s_k - s_i \right). \qquad (12)$$

Having $s_k \geq s_i$ by the selection of *k*, the scalars α_i are all positive, hence all elements of $\tilde{D}$ are positive. By adding at most *N*-1 *Cross Structured* matrices we get a symmetric matrix where all rows have the same sum of elements, resulting in $\tilde{D} \cdot \underline{1} = \omega_0 \underline{1}$. We shall refer to $\tilde{D}$ as the *Weighted Distance-Matrix*. The channel distortion is now:

$$D_C = \frac{1}{2} trace\left\{ Q\pi^T \tilde{D} \pi \right\} - S \qquad (13)$$

where

$$S = \sum_{i=0}^{N-1} \alpha_i . \qquad (14)$$

As this point, it is interesting to note that both the channel matrix Q and the Weighted Distance-Matrix $\tilde{D}$ are symmetric, have nonnegative entries, and have the vector $\underline{1} = [1 \;\; 1 \;\; \cdots \;\; 1]^T$ as an eigenvector. Moreover, all eigenvalues of both matrices are real.

It is shown in [2] that the eigenvalue 1 of the matrix Q and the eigenvalue $\omega_0 > 0$ of the matrix $\tilde{D}$, both corresponding to the eigenvector $\underline{1}$, are the largest in absolute value for each corresponding matrix.

We now perform a unitary diagonalization on both matrices:

$$\begin{aligned} Q &= V \cdot \Lambda \cdot V^T, \quad V \cdot V^T = I \\ \tilde{D} &= W \cdot \Omega \cdot W^T, \quad W \cdot W^T = I \end{aligned} \qquad (15)$$

We define $\lambda_i = \Lambda_{ii}, \omega_i = \Omega_{ii}, i = 0,1,\ldots,N-1$ and, without loss of generality, we sort the eigenvalues (and corresponding eigenvectors) in Λ and Ω in decreasing order - λ_0 (ω_0) is the largest eigenvalues of the matrix Q ($\tilde{D}$).

It is shown then in [2] that the Channel Distortion for all possible index assignments is bounded by:

$$\frac{1}{2}\left(\lambda_0 \omega_0 + \sum_{i=1}^{N-1} \lambda_i \omega_{N-i} \right) - S \leq D_C \leq \frac{1}{2}\left(\lambda_0 \omega_0 + \sum_{i=1}^{N-1} \lambda_i \omega_i \right) - S \qquad (16)$$

It is also shown that the average value of the channel distortion over all possible index assignments is:

$$\langle D_C \rangle = \frac{1}{2} \lambda_0 \omega_0 + \frac{1}{2(N-1)} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \lambda_i \omega_j - S \qquad (17)$$

In conclusion, in order to find the desired bounds and the value of $\langle D_C \rangle$ for given Channel Transition Matrix - *Q*, VQ distance matrix – *D*, and a-priori probabilities matrix – *P*, one should carry out the following steps:

1. Calculate the scalar S and the *Weighted Distance Matrix*, $\tilde{D}$, using (10), (11), (12) and (14).
2. Calculate the eigenvalues of the *Channel Matrix* Q and of the *Weighted Distance Matrix* $\tilde{D}$. Sort the eigenvalues in decreasing order.
3. Calculate the upper and lower bounds using (16) and the average performance using (17).

4. SIMULATION RESULTS

In the following, the lower and upper bounds are compared with the average performance as well as with assignments that were obtained in simulations. We used the sub-optimal index-switching algorithm [8] to obtain "good" and "poor" simulation index assignments (IA). After selecting an initial assignment, we randomly exchange indices of codevectors. When searching for the good (poor) simulation assignment, the new assignment is kept if it performs better (worse) than its predecessor.

4.1 A 4 bit Uniform Quantizer and a Uniform Source under the BSC

We examine the bounds and the average performance (over all index assignments) for a 4-bit uniform scalar quantizer, a uniform source and a Binary Symmetric Channel. Two sets of graphs are shown in Fig. 2, in the first set the 4 bits are sent through a BSC, and in the second set they are protected against single-bit error using a (7,4) Hamming Error Correcting Code (ECC) [9]. It can be seen that the slope of the lines is roughly 10dB/decade for the BSC and 20dB/decade for the ECC case. It is shown in [2] that the distortion due to the Natural Binary Code (NBC) coincides in this case with the lower bound. The upper bound is about 0.5dB higher than the worst assignment that was found in simulations. The ratio between the upper and lower bounds is approximately 6dB for the BSC case and 3.6dB for the ECC. The implementation of the channel protection brought the bounds closer together, decreasing the importance of index assignment.

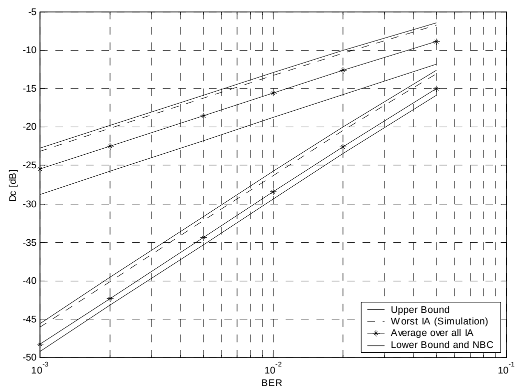

Fig. 2 – Upper and lower bounds and the average channel-distortion over all possible index assignments of a 4-bit Uniform Scalar Quantizer and a uniform source under the BSC and under the BSC and (7,4) Hamming ECC.

4.2 Three-dimensional, 8-bit PDF-optimized Vector Quantizer for palette limited Images using the L*a*b* color space

The L*a*b* color space was developed by the CIE [10] in order to better represent human vision perception. Pixels' colors are organized in three componenet: An achromatic component L*, and two chroamtic ones: a* and b*. The correlation between the L*a*b* space and the human perception justifies the use of an Euclidean distance measure. We examine here an 8-bit L*a*b* Vector Quantizer from [11]. The bounds are shown in Fig. 3.

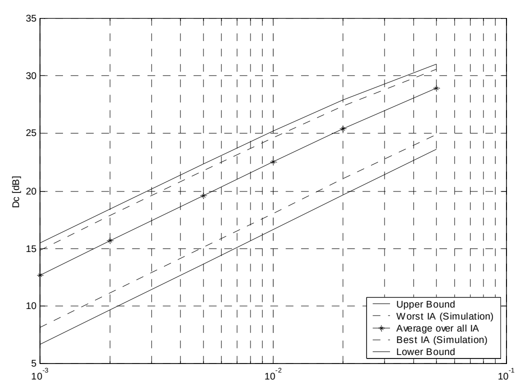

Fig. 3 – Upper and lower bounds and the average channel-distortion over all possible index assignments of an 8-bit L*a*b*-space Image Vector Quantizer under the BSC.

The Upper Bound is about 0.6dB higher than channel distortion due to the worst possible index assignment. The Lower Bound is 1.5dB lower than the distortion for the Best assignment. The ratio between the Upper and Lower bounds is 8.8dB, suggesting that a significant performance gain may be achieved by a good index assignment.

5. CONCLUSIONS

In this paper we have presented upper and lower bounds (and a related expression for the average performance) of the distortion due to channel errors for Vector Quantizers, over all possible index assignments. These results enable the VQ designer to estimate the gain that may be obtained by a search for an efficient index assignment and to estimate the performance of a given index assignment.

Numerical examples were given for the Binary Symmetric Channel, with and without a channel Error Correcting Code. The bounds were compared with simulation results that searched for "good" and "poor" assignments using the sub-optimal index-switching algorithm. The bounds are reasonably close to the performance of the assignments found in simulations. Nevertheless, the tightness characteristic of the proposed bounds is left to further study. More simulation were performed, supporting the shown results, but are not shown here because of lack of space.

Utilization of an Error Correcting Code decreases the gap between the lower and the upper bounds. This result agrees with the intuition that channel protection reduces the importance of index assignment.

6. REFERENCES

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