2-D SINUSOIDAL AMPLITUDE ESTIMATION WITH APPLICATION TO 2-D SYSTEM IDENTIFICATION

Hongbin Li[1], *Wei Sun*[1], *Petre Stoica*[2], *Jian Li*[3]

[1]Department of Electrical and Computer Engineering, Stevens Institute of Technology, Hoboken, NJ 07030, USA.
[2]Department of Systems and Control, Uppsala University, P.O.Box 27, SE–751 03, Uppsala, Sweden.
[3]Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611, USA.

ABSTRACT

In [1], we studied amplitude estimation of one-dimensional (1-D) sinusoidal signals from measurements corrupted by possibly colored observation noise. We herein extend the results for two-dimensional (2-D) amplitude estimation. In particular, we investigate the 2-D sinusoidal amplitude estimation within the general frameworks of least squares (LS), weighted least squares (WLS), and MAtched FIlterbank (MAFI) estimation. A variety of 2-D amplitude estimators are presented, which are all asymptotically statistically efficient. The performances of these estimators in finite samples are compared numerically with one another. Making use of amplitude estimation techniques, we introduce a new scheme for 2-D system identification, which is shown to be computationally simpler and statistically more accurate than the conventional output error method (OEM), when the observation noise is colored.

1. INTRODUCTION

Consider the noise-corrupted observation of K 2-D sinusoids:

$$x(n,\bar{n}) = \sum_{k=1}^{K} \alpha_k e^{j2\pi(f_k n + \bar{f}_k \bar{n})} + v(n,\bar{n}), \quad n = 0,\cdots,N-1;\ \bar{n} = 0,\cdots,\bar{N}-1, \tag{1}$$

where $\{\alpha_k\}_{k=1}^K$ is the complex amplitude of the k-th 2-D sinusoid at the 2-D frequency pair $(f_k,\bar{f}_k)$, and $v(n,\bar{n})$ is the 2-D complex-valued additive observation noise assumed to be stationary with zero-mean and unknown finite power spectral density (PSD) $\phi(f,\bar{f})$. We assume that $\{f_k,\bar{f}_k\}_{k=1}^K$ are known and distinct from one another. The problem of interest is to estimate the 2-D amplitudes $\{\alpha_k\}_{k=1}^K$ from the observations $\{x(n,\bar{n})\}$. With matrix notations, we can express (1) as

$$\mathbf{X} = \mathbf{A}\boldsymbol{\Lambda}\bar{\mathbf{A}}^T + \mathbf{V}, \tag{2}$$

where $\mathbf{X} \in \mathbb{C}^{N\times\bar{N}}$, with $n\bar{n}$-th element given by $x(n,\bar{n})$, $\mathbf{V} \in \mathbb{C}^{N\times\bar{N}}$ is similarly formed from $\{v(n,\bar{n})\}$, and $\boldsymbol{\Lambda} \triangleq \text{diag}\{\alpha_1 \cdots \alpha_K\}$. $\mathbf{A}$ and $\bar{\mathbf{A}}$ are $N\times K$ and $\bar{N}\times K$ Vandermonde matrices with the k-th column given by $\boldsymbol{a}(f_k) = [1\ e^{j2\pi f_k}\ \cdots\ e^{j(N-1)2\pi f_k}]^T$ and $\bar{\boldsymbol{a}}(\bar{f}_k) = [1\ e^{j2\pi\bar{f}_k}\ \cdots\ e^{j(\bar{N}-1)2\pi\bar{f}_k}]^T$, respectively. Vectorizing both sides of (2) yields:

$$\boldsymbol{x} = (\bar{\mathbf{A}} \diamond \mathbf{A})\boldsymbol{\alpha} + \boldsymbol{v} \triangleq \boldsymbol{\Psi}\boldsymbol{\alpha} + \boldsymbol{v}, \tag{3}$$

where $\boldsymbol{x} \triangleq \text{vec}\{\mathbf{X}\}$, $\boldsymbol{v} \triangleq \text{vec}\{\mathbf{V}\}$, $\boldsymbol{\alpha} \triangleq [\alpha_1 \cdots \alpha_K]^T$, and $\diamond$ is the Khatri-Rao product, which is a column-wise Kronecker product [2].

THIS WORK WAS SUPPORTED IN PART BY THE SENIOR INDIVIDUAL GRANT PROGRAM OF THE SWEDISH FOUNDATION FOR STRATEGIC RESEARCH AND THE NATIONAL SCIENCE FOUNDATION GRANT MIP-9457388.

In this paper, we present a number of 2-D sinusoidal amplitude estimation algorithms within the frameworks of LS, WLS, and MAFI estimation approaches. Making use of 2-D amplitude estimation techniques, we propose a new approach which has a simple closed-form solution to the 2-D system identification problem. The following notational conventions are used to distinguish among various amplitude estimators. For example, LSE $(1,0,1)$ denotes the LS estimator that uses a single data "snapshot", employs no pre-filtering, and estimates one amplitude at a time. Likewise, MAFI$(L\bar{L},K,K)$ denotes the MAFI estimator that splits the data into $L\bar{L}$ submatrices, utilizes a bank of K pre-filters, and estimates K amplitudes simultaneously. The remaining amplitude estimators are similarly designated.

2. LS AMPLITUDE ESTIMATORS

2.1. LSE$(1,0,K)$

From (3), the LS estimate of $\boldsymbol{\alpha}$ is readily obtained as

$$\hat{\boldsymbol{\alpha}} = (\boldsymbol{\Psi}^H\boldsymbol{\Psi})^{-1}\boldsymbol{\Psi}^H\boldsymbol{x}, \tag{4}$$

where $(\cdot)^H$ denotes the conjugate transpose. It can be shown that LSE$(1,0,K)$ is unbiased. In general, LSE$(1,0,K)$ is statistically inefficient with colored noise, but is asymptotically (for large samples) statistically efficient [3].

2.2. LSE$(1,0,1)$

Treating $(K-1)$ sinusoids in (1) as noise, we obtain the LSE $(1,0,1)$ estimator which estimates only one amplitude at a time:

$$\hat{\alpha}_k = \frac{1}{N\bar{N}}\sum_{n=0}^{N-1}\sum_{\bar{n}=0}^{\bar{N}-1} x(n,\bar{n})e^{-j2\pi(f_k n + \bar{f}_k\bar{n})},\ k = 1,\cdots,K. \tag{5}$$

The above estimator is the normalized 2-D discrete Fourier transform (DFT) of $x(n,\bar{n})$ at frequency $(f_k,\bar{f}_k)$. LSE $(1,0,1)$ is biased but asymptotically unbiased [3]. Similar to LSE$(1,0,K)$, LSE$(1,0,1)$ is statistically inefficient for colored noise, but is asymptotically statistically efficient [3]. Due to its computational simplicity, LSE$(1,0,1)$ is usually preferred when the number of data samples is relatively large .

3. WLS AMPLITUDE ESTIMATORS

3.1. WLSE$(L\bar{L},0,K)$

Let $\mathbf{X}_{l,\bar{l}} = \{x(k,\bar{k}),\ k = l,\cdots,l+M-1;\ \bar{k} = \bar{l},\cdots,\bar{l}+\bar{M}-1\} \in \mathbb{C}^{M\times\bar{M}}$, $l = 0,\cdots,L-1;\ \bar{l} = 0,\cdots,\bar{L}-1$, where $L \triangleq N-M+1$, $\bar{L} \triangleq \bar{N}-\bar{M}+1$, and $\mathbf{V}_{l,\bar{l}} \in \mathbb{C}^{M\times\bar{M}}$ be similarly formed from $\{v(n,\bar{n})\}$. $\mathbf{X}_{l,\bar{l}}$ can be expressed as

$$\mathbf{X}_{l,\bar{l}} = \mathbf{A}_M\boldsymbol{\Lambda}_{l,\bar{l}}\bar{\mathbf{A}}_{\bar{M}}^T + \mathbf{V}_{l,\bar{l}}, \tag{6}$$

where $\mathbf{A}_M \triangleq [\boldsymbol{a}_M(f_1)\ \cdots\ \boldsymbol{a}_M(f_K)]$ is a Vandermonde matrix with $\boldsymbol{a}_M(f_k) \triangleq [1\ e^{j2\pi f_k}\ \ldots\ e^{j(M-1)2\pi f_k}]^T$, $\bar{\mathbf{A}}_{\bar{M}}$ and $\bar{\boldsymbol{a}}_{\bar{M}}(\bar{f}_k)$ are defined similarly to $\mathbf{A}_M$ and $\boldsymbol{a}_M(f_k)$, respectively, and $\boldsymbol{\Lambda}_{l,\bar{l}} \triangleq \boldsymbol{\Lambda}\ \text{diag}\{e^{j2\pi(f_1 l+\bar{f}_1\bar{l})}\ \ldots\ e^{j2\pi(f_K l+\bar{f}_K\bar{l})}\} \triangleq \boldsymbol{\Lambda}\boldsymbol{\Omega}_{l,\bar{l}}$. Let $\boldsymbol{x}_{l,\bar{l}} = \text{vec}\{\mathbf{X}_{l,\bar{l}}\}$ and $\boldsymbol{v}_{l,\bar{l}} = \text{vec}\{\mathbf{V}_{l,\bar{l}}\}$. Vectorizing (6) yields:

$$\boldsymbol{x}_{l,\bar{l}} \triangleq \boldsymbol{\Psi}_{M\bar{M}}\boldsymbol{\Omega}_{l,\bar{l}}\boldsymbol{\alpha} + \boldsymbol{v}_{l,\bar{l}}, \tag{7}$$

where $\boldsymbol{\Psi}_{M\bar{M}} \triangleq [\bar{\boldsymbol{a}}_{\bar{M}}(\bar{f}_1)\otimes\boldsymbol{a}_M(f_1)\ \ldots\ \bar{\boldsymbol{a}}_{\bar{M}}(\bar{f}_K)\otimes\boldsymbol{a}_M(f_K)]$.

The WLS (Markov-like) estimate of $\boldsymbol{\alpha}$ is given by [4]

$$\begin{aligned}\hat{\boldsymbol{\alpha}} &= \left(\sum_{l=0}^{L-1}\sum_{\bar{l}=0}^{\bar{L}-1}\boldsymbol{\Omega}_{l,\bar{l}}^H\boldsymbol{\Psi}_{M\bar{M}}^H\hat{\mathbf{Q}}^{-1}\boldsymbol{\Psi}_{M\bar{M}}\boldsymbol{\Omega}_{l,\bar{l}}\right)^{-1}\\ &\quad\times\left(\sum_{l=0}^{L-1}\sum_{\bar{l}=0}^{\bar{L}-1}\boldsymbol{\Omega}_{l,\bar{l}}^H\boldsymbol{\Psi}_{M\bar{M}}^H\hat{\mathbf{Q}}^{-1}\boldsymbol{x}_{l,\bar{l}}\right),\end{aligned} \tag{8}$$

where $\hat{\mathbf{Q}}$ is an estimate of $\mathbf{Q} \triangleq E\{\boldsymbol{v}_{l,\bar{l}}\boldsymbol{v}_{l,\bar{l}}^H\}$. An estimate of $\mathbf{Q}$ is obtained as follows. Assume that the initial phases of the sinusoids are independently, identically and uniformly distributed over $[-\pi,\pi)$, and are independent of the noise term $\mathbf{v}_{l,\bar{l}}$ in (7). Then,

$$\mathbf{R} \triangleq E\{\boldsymbol{x}_{l,\bar{l}}\boldsymbol{x}_{l,\bar{l}}^H\} = \boldsymbol{\Psi}_{M\bar{M}}\mathbf{P}\boldsymbol{\Psi}_{M\bar{M}}^H + \mathbf{Q}, \tag{9}$$

where $\mathbf{P} = \text{diag}\{|\alpha_1^2|\ \cdots\ |\alpha_K^2|\}$. Hence, a straightforward estimate of $\mathbf{Q}$ is obtained as

$$\hat{\mathbf{Q}} = \hat{\mathbf{R}} - \boldsymbol{\Psi}_{M\bar{M}}\hat{\mathbf{P}}\boldsymbol{\Psi}_{M\bar{M}}^H, \tag{10}$$

where $\hat{\mathbf{R}} \triangleq \frac{1}{L\bar{L}}\sum_{l=0}^{L-1}\sum_{\bar{l}=0}^{\bar{L}-1}\boldsymbol{x}_{l,\bar{l}}\boldsymbol{x}_{l,\bar{l}}^H$ is the sample covariance of $\boldsymbol{x}_{l,\bar{l}}$ and $\hat{\mathbf{P}}$ is some initial estimate of $\mathbf{P}$.

To eliminate the need for an initial estimate $\hat{\mathbf{P}}$, we note that for sufficiently large $N, \bar{N}, M$, and $\bar{M}$ [3],

$$\hat{\mathbf{Q}}^{-1}\boldsymbol{\Psi}_{M\bar{M}}\boldsymbol{\Omega}_{l,\bar{l}} \approx \hat{\mathbf{R}}^{-1}\boldsymbol{\Psi}_{M\bar{M}}\boldsymbol{\Omega}_{l,\bar{l}}\boldsymbol{\Sigma}, \tag{11}$$

where $\boldsymbol{\Sigma} \triangleq \hat{\mathbf{P}}\boldsymbol{\Psi}_{M\bar{M}}^H\hat{\mathbf{Q}}^{-1}\boldsymbol{\Psi}_{M\bar{M}} + \mathbf{I}_K$. Therefore, the estimator in (8) becomes

$$\begin{aligned}\hat{\boldsymbol{\alpha}} &= \left(\sum_{l=0}^{L-1}\sum_{\bar{l}=0}^{\bar{L}-1}\boldsymbol{\Omega}_{l,\bar{l}}^H\boldsymbol{\Psi}_{M\bar{M}}^H\hat{\mathbf{R}}^{-1}\boldsymbol{\Psi}_{M\bar{M}}\boldsymbol{\Omega}_{l,\bar{l}}\right)^{-1}\\ &\quad\times\left(\sum_{l=0}^{L-1}\sum_{\bar{l}=0}^{\bar{L}-1}\boldsymbol{\Omega}_{l,\bar{l}}^H\boldsymbol{\Psi}_{M\bar{M}}^H\hat{\mathbf{R}}^{-1}\boldsymbol{x}_{l,\bar{l}}\right),\end{aligned} \tag{12}$$

which is an extension of the 2-D Capon spectral estimator (see, e.g., [5], [6]) to multiple sinusoids.

An alternative estimate of $\mathbf{Q}$, other than (10), can be obtained as described next. Let $\tilde{\boldsymbol{a}}_{M\bar{M}}(f_k,\bar{f}_k) \triangleq \bar{\boldsymbol{a}}_{\bar{M}}(\bar{f}_k)\otimes\boldsymbol{a}_M(f_k)$. Then (7) can be expressed as

$$\boldsymbol{x}_{l,\bar{l}} = \sum_{k=1}^{K}\alpha_k\tilde{\boldsymbol{a}}_{M\bar{M}}(f_k,\bar{f}_k)e^{j2\pi(f_k l+\bar{f}_k\bar{l})} + \boldsymbol{v}_{l,\bar{l}}. \tag{13}$$

The LS estimate of $\alpha_k\tilde{\boldsymbol{a}}_{M\bar{M}}(f_k,\bar{f}_k)$ from (13) is

$$\widehat{\alpha_k\tilde{\boldsymbol{a}}_{M\bar{M}}(f_k,\bar{f}_k)} = \frac{1}{L\bar{L}}\sum_{l=0}^{L-1}\sum_{\bar{l}=0}^{\bar{L}-1}\boldsymbol{x}_{l,\bar{l}}e^{-j2\pi(f_k l+\bar{f}_k\bar{l})} \triangleq \boldsymbol{\xi}_k. \tag{14}$$

It follows from (14) and (9) that a new estimate of $\mathbf{Q}$ is

$$\hat{\mathbf{Q}} = \hat{\mathbf{R}} - \sum_{k=1}^{K}\boldsymbol{\xi}_k\boldsymbol{\xi}_k^H. \tag{15}$$

The WLSE$(L\bar{L},0,K)$ that uses (8) with $\hat{\mathbf{Q}}$ given in (15) does not require any initial amplitude estimates. It is an extension of the 2-D APES algorithm [5], [6] to multiple sinusoids with known frequencies.

3.2. WLSE$(L\bar{L},0,1)$

If we apply the WLS technique as described above but restrict it to estimate one sinusoid at a time, then the WLSE$(L\bar{L},0,K)$ amplitude estimator reduces to WLSE$(L\bar{L},0,1)$. In particular, the WLSE$(L\bar{L},0,1)$ estimator using (8) and (10) is given by

$$\hat{\alpha}_k = \frac{\tilde{\boldsymbol{a}}_{M\bar{M}}^H(f_k,\bar{f}_k)\hat{\mathbf{R}}^{-1}\boldsymbol{\xi}_k}{\tilde{\boldsymbol{a}}_{M,\bar{M}}^H(f_k,\bar{f}_k)\hat{\mathbf{R}}^{-1}\tilde{\boldsymbol{a}}_{M,\bar{M}}(f_k,\bar{f}_k)},\ k=1,\cdots,K, \tag{16}$$

whereas the WLSE$(L\bar{L},0,1)$ estimator using (8) and (15) is

$$\hat{\alpha}_k = \frac{\tilde{\boldsymbol{a}}_{M\bar{M}}^H(f_k,\bar{f}_k)[\hat{\mathbf{R}}-\boldsymbol{\xi}_k\boldsymbol{\xi}_k^H]^{-1}\boldsymbol{\xi}_k}{\tilde{\boldsymbol{a}}_{M,\bar{M}}^H(f_k,\bar{f}_k)[\hat{\mathbf{R}}-\boldsymbol{\xi}_k\boldsymbol{\xi}_k^H]^{-1}\tilde{\boldsymbol{a}}_{M,\bar{M}}(f_k,\bar{f}_k)},\quad k=1,\cdots,K. \tag{17}$$

It should be noted that, unlike (12), (16) is *exactly* equivalent to using (10) with (8). The amplitude estimators (16) and (17) have the same form as the 2-D Capon and APES spectral estimators [5], [6], [7]. It was shown in [6] that though both estimators are asymptotically efficient, they behave quite differently with finite samples. Specifically, (16) is *biased downward*, whereas (17) is *unbiased* within a second-order approximation.

4. MAFI AMPLITUDE ESTIMATORS

Let $\mathbf{H}^H \in \mathbb{C}^{K\times M\bar{M}}$ be such that each row of which corresponds to an $M\bar{M}$-tap FIR filter whose center frequencies correspond to the 2-D frequencies of one sinusoid of interest. Applying $\mathbf{H}^H$ to both sides of (7), we obtain the filterbank output as

$$\boldsymbol{y}_{l,\bar{l}} \triangleq \mathbf{H}^H\boldsymbol{x}_{l,\bar{l}} = \mathbf{H}^H\boldsymbol{\Psi}_{M\bar{M}}\boldsymbol{\Omega}_{l,\bar{l}}\boldsymbol{\alpha} + \mathbf{H}^H\boldsymbol{v}_{l,\bar{l}}. \tag{18}$$

The MAFI approach chooses $\mathbf{H}^H$ to maximize the output SNR:

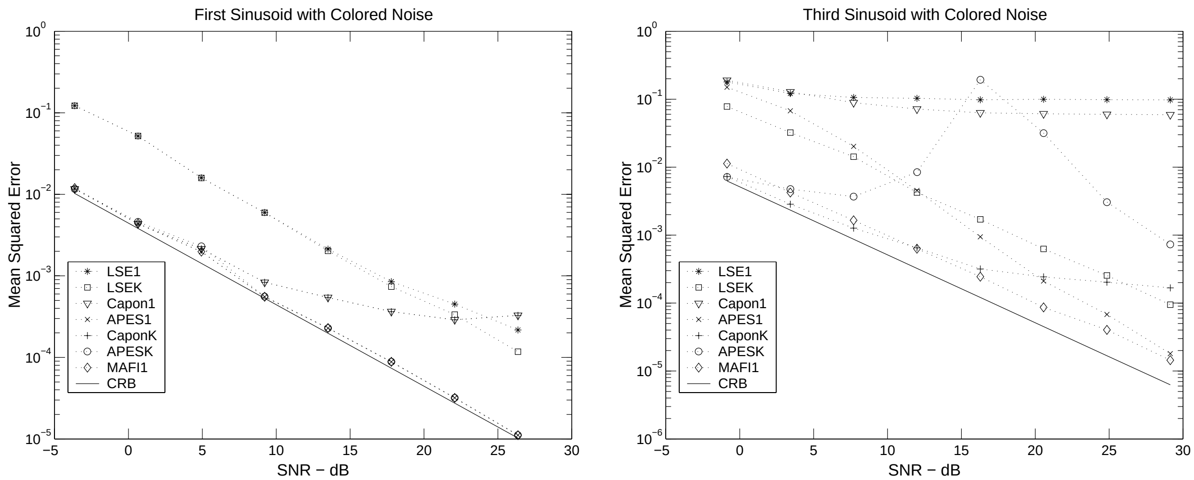
$$\mathbf{H} = \arg\max_{\mathbf{H}}\underbrace{\text{tr}\left\{(\mathbf{H}^H\hat{\mathbf{Q}}\mathbf{H})^{-1}\mathbf{H}^H(\boldsymbol{\Psi}_{M\bar{M}}\hat{\mathbf{P}}\boldsymbol{\Psi}_{M\bar{M}}^H)\mathbf{H}\right\}}_{\text{SNR}}. \tag{19}$$

The solution to (19) is *not unique* [1]. One that has a simple closed-form is $\mathbf{H} = \hat{\mathbf{Q}}^{-1}\boldsymbol{\Psi}_{M\bar{M}}(\boldsymbol{\Psi}_{M\bar{M}}^H\hat{\mathbf{Q}}^{-1}\boldsymbol{\Psi}_{M\bar{M}})^{-1}$ [1]. It is easily verified that the so obtained $\mathbf{H}$ satisfies $\mathbf{H}^H\boldsymbol{\Psi}_{M\bar{M}} = \mathbf{I}_K$, which implies that each FIR filter in $\mathbf{H}^H$ passes the sinusoid of interest undistorted while completely eliminates the other sinusoids. As such, (18) reduces to

$$\boldsymbol{y}_{l,\bar{l}} = \boldsymbol{\Omega}_{l,\bar{l}}\boldsymbol{\alpha} + \mathbf{H}^H\boldsymbol{v}_{l,\bar{l}}. \tag{20}$$

Observe that the covariance matrix of $\mathbf{H}^H\boldsymbol{v}_{l,\bar{l}}$ is given by $\mathbf{H}^H\hat{\mathbf{Q}}\mathbf{H} = (\boldsymbol{\Psi}_{M\bar{M}}^H\hat{\mathbf{Q}}^{-1}\boldsymbol{\Psi}_{M\bar{M}})^{-1}$. The MAFI estimate is obtained by using the WLS (Markov-like) technique to (20) [3]:

$$\begin{aligned}\hat{\boldsymbol{\alpha}} &= \left(\sum_{l=0}^{L-1}\sum_{\bar{l}=0}^{\bar{L}-1}\boldsymbol{\Omega}_{l,\bar{l}}^H\boldsymbol{\Psi}_{M\bar{M}}^H\hat{\mathbf{Q}}^{-1}\boldsymbol{\Psi}_{M\bar{M}}\boldsymbol{\Omega}_{l,\bar{l}}\right)^{-1}\\ &\quad\times\left(\sum_{l=0}^{L-1}\sum_{\bar{l}=0}^{\bar{L}-1}\boldsymbol{\Omega}_{l,\bar{l}}^H\boldsymbol{\Psi}_{M\bar{M}}^H\hat{\mathbf{Q}}^{-1}\boldsymbol{x}_{l,\bar{l}}\right),\end{aligned} \tag{21}$$



(a) (b)

Fig. 1. Empirical MSE and CRB vs. SNR with $N = \bar{N} = 16$, $M = \bar{M} = 4$. (a) α_1. (b) α_3.

which coincides with (12). Note that the MAFI$(L\bar{L}, K, K)$ approach is more general since it includes WLSE$(L\bar{L}, 0, K)$ as a special case and other MAFI estimators without a WLS interpretation exist [3]. We next derive such a MAFI estimator.

Let $y_{l,\bar{l}}(k)$ and $\nu_{l,\bar{l}}(k)$ be the k-th element of $\boldsymbol{y}_{l,\bar{l}}$ and $\mathbf{H}^H \boldsymbol{v}_{l,\bar{l}}$, respectively. Then from (20) we have

$$y_{l,\bar{l}}(k) = \alpha_k e^{j2\pi(f_k l + \bar{f}_k \bar{l})} + \nu_{l,\bar{l}}(k),\ k = 1, \ldots, K. \qquad (22)$$

Applying LS to (22) gives the MAFI$(L\bar{L}, K, 1)$ estimator:

$$\hat{\alpha}_k = \frac{1}{L\bar{L}} \sum_{l=0}^{L-1} \sum_{\bar{l}=0}^{\bar{L}-1} y_{l,\bar{l}}(k) e^{-j2\pi(f_k l + \bar{f}_k \bar{l})},\ k = 1, \cdots, K. \quad (23)$$

Note that, unlike the other one-at-a-time estimators, MAFI$(L\bar{L}, K, 1)$ requires knowledge of the number and frequencies of the sinusoids, owing to the need to design the filterbank.

5. NUMERICAL EXAMPLE

For notational simplicity, we adopt the following acronyms for the 2-D amplitude estimators: **i**. LSE1: using (5); **ii**. LSEK: using (4); **iii**. Capon1: using (16); **iv**. APES1: using (17); **v**. CaponK: using (12); **vi**. APESK: using (8) along with (15); and **vii**. MAFI1: using (23) along with (15). The data consists of $K = 3$ 2-D complex sinusoids in a zero-mean complex Gaussian noise. The frequencies of the sinusoids are $(0.45, 0.35)$, $(0.235, 0.135)$, and $(0.2, 0.1)$, and the amplitudes are $\alpha_1 = e^{j\pi/4}$, $\alpha_2 = e^{j\pi/3}$, and $\alpha_3 = e^{j\pi/4}$, respectively. The colored noise is a 2-D autoregressive (AR) process: $v(n, \bar{n}) = 0.99 v(n-1, \bar{n}-1) + e(n, \bar{n})$, where $e(n, \bar{n})$ is a complex-valued white Gaussian noise with zero-mean and variance $\sigma^2 = 0.01$. The SNR of the k-th sinusoid is defined as $\text{SNR}_k = 10 \log_{10} \frac{|\alpha_k|^2}{\phi(f_k, \bar{f}_k)}$. [8]. Consider $N = \bar{N} = 16$, and for the WLS and MAFI estimators, $M = \bar{M} = 4$. Figure 1(a) shows the mean squared errors (MSE) of the seven estimators for α_1 as the SNR changes. We see that APES1, APESK, and MAFI1 are close to the Cramér-Rao bound (CRB) [3], while both LS estimators are away from the CRB. CaponK is away from the CRB at high SNRs, which is due to a bias introduced in the approximation of (11) [1]. Capon1 is also biased for finite samples [6], which causes it to deviate from the CRB at high SNRs. The performances of the above estimators are somewhat different when some sinusoids are close to the one of interest, which is the case shown in Figure 1(b) for α_3. In particular, LSE1, Capon1, and APES1 degrade considerably while APES1 still appears to be consistent. The performance of APESK becomes quite sensitive for certain SNRs. In the current case, MAFI1 appears to be the best estimator.

6. SYSTEM IDENTIFICATION

Consider the following 2-D linear discrete-time system:

$$\begin{aligned} x(n, \bar{n}) &= H(z^{-1}, \bar{z}^{-1}) u(n, \bar{n}) + v(n, \bar{n}), \\ n &= 0, \cdots, N-1;\ \bar{n} = 0, \cdots, \bar{N}-1, \end{aligned} \qquad (24)$$

where the probing signal is $u(n, \bar{n}) = \sum_{k=1}^{K} \gamma_k e^{j2\pi(f_k n + \bar{f}_k \bar{n})}$, $v(n, \bar{n})$ is the measurement noise, and

$$H(z^{-1}, \bar{z}^{-1}) = \frac{B(z^{-1}, \bar{z}^{-1})}{A(z^{-1}, \bar{z}^{-1})} = \frac{\sum_{i=0}^{r-1} \sum_{j=0}^{s-1} b_{i,j} z^{-i} \bar{z}^{-j}}{\sum_{i=0}^{p-1} \sum_{j=0}^{q-1} a_{i,j} z^{-i} \bar{z}^{-j}},$$

where $a_{0,0} = 1$, $b_{0,0} = 0$, and $(z^{-1}, \bar{z}^{-1})$ are the unit-delay operators. Assume that the system orders $r,\ s,\ p$, and q are known, and $K \geq (pq-1) + (rs-1)$. Let $\boldsymbol{a} \triangleq [a_{0,1}\ \cdots\ a_{0,q-1}\ \cdots\ a_{p-1,q-1}]^T$ and $\boldsymbol{b} \triangleq [b_{0,1}\ \cdots\ b_{0,s-1} \cdots\ b_{r-1,s-1}]^T$. The 2-D system identification problem is to estimate the system parameters $\boldsymbol{a}$ and $\boldsymbol{b}$ from the outputs $\{x(n, \bar{n})\}$.

The output error method (OEM) solves the above problem by minimizing the nonlinear cost function [4]

$$C_{\text{OEM}}(\boldsymbol{a}, \boldsymbol{b}) = \sum_{n=0}^{N-1} \sum_{\bar{n}=0}^{\bar{N}-1} \left| x(n, \bar{n}) - H(z^{-1}, \bar{z}^{-1}) u(n, \bar{n}) \right|^2.$$

Define $\alpha_k(\boldsymbol{a}, \boldsymbol{b}) \triangleq \gamma_k H(e^{-j2\pi f_k}, e^{-j2\pi \bar{f}_k})$. For sufficiently large N and $\bar{N}$ such that the transient response in the output can be ignored, (24) can be approximated as

$$x(n, \bar{n}) = \sum_{k=1}^{K} \alpha_k(\boldsymbol{a}, \boldsymbol{b}) e^{j2\pi(f_k n + \bar{f}_k \bar{n})} + v(n, \bar{n}), \qquad (25)$$

from which we can estimate $\{\alpha_k(\boldsymbol{a}, \boldsymbol{b})\}_{k=1}^{K}$ by using any amplitude estimator discussed before. Once the amplitude estimates of

$\{\alpha_k(\boldsymbol{a},\boldsymbol{b})\}_{k=1}^{K}$ are available, we select the $(pq-1)+(rs-1)$ largest ones (in magnitude) out of the K amplitude estimates and denote them by $\hat{\alpha}_k,\ k=1,\ldots,(pq-1)+(rs-1)$. Choose $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\hat{\alpha}_k = \alpha_k(\boldsymbol{a},\boldsymbol{b}),\ k=1,\cdots,(pq-1)+(rs-1)$. Or, equivalently,

$$\hat{\alpha}_k A(e^{-j2\pi f_k}, e^{-j2\pi \bar{f}_k}) = \gamma_k B(e^{-j2\pi f_k}, e^{-j2\pi \bar{f}_k}), \quad k=1,\cdots,(pq-1)+(rs-1), \tag{26}$$

which is a set of $(pq-1)+(rs-1)$ linear equations with $(pq-1)+(rs-1)$ unknowns. Solving these equations gives an estimate of the system parameters $\boldsymbol{a}$ and $\boldsymbol{b}$.

Now consider a system identification example that involves colored measurement noise. The system is

$$\begin{aligned} A(z^{-1},\bar{z}^{-1}) &= 1 - j0.8\bar{z}^{-1} - (0.529 + j0.7281)z^{-1} \\ &\quad -(0.5825 - j0.4232)z^{-1}\bar{z}^{-1}, \\ B(z^{-1},\bar{z}^{-1}) &= 1 + (0.2014 - j0.7846)\bar{z}^{-1} \\ &\quad -(0.2194 + j0.6753)z^{-1} - (0.574 - j0.0361)z^{-1}\bar{z}^{-1}. \end{aligned}$$

The probing signal consists of $K=8$ 2-D complex sinusoids at frequencies: $(-0.45,\ 0.48)$, $(-0.317,\ 0.347)$, $(-0.183,\ 0.213)$, $(-0.05, 0.08)$, $(0.05, -0.08)$, $(0.183, -0.213)$, $(0.317, -0.347)$, and $(0.45,\ -0.48)$. The noise $v(n,\bar{n})$ is an AR process, similarly generated as in Section 5. The performance criteria considered here are the averaged root mean squared errors (ARMSE) of the parameter estimates and the number of flops associated with each method. The ARMSE for the $\boldsymbol{a}$ parameters is defined as $\mathrm{ARMSE}\{\hat{\boldsymbol{a}}\} = \frac{1}{pq-1}\sum_{i,j,(i,j)\neq(0,0)} \mathrm{RMSE}\{\hat{a}_{i,j}\}$. The ARMSE for the $\boldsymbol{b}$ parameters is similarly defined. The results are shown in Figures 2(a) to 2(c). We can see that APES1 and MAFI1 not only obtain statistically more accurate parameter estimates than OEM does, but they are also computationally simpler than the latter.

7. REFERENCES

[1] P. Stoica, H. Li, and J. Li, “Amplitude estimation of sinusoidal signals: Survey, new results, and an application,” *IEEE Transactions on Signal Processing*, vol. 48, pp. 338–352, February 2000.

[2] J. W. Brewer, “Kronecker products and matrix calculus in system theory,” *IEEE Transactions on Circuits and Systems*, vol. CAS-25, no. 9, pp. 772–781, September 1978.

[3] H. Li, W. Sun, P. Stoica, and J. Li, “2-D sinusoidal amplitude estimation with application to 2-D system identification,” *IEEE Transactions on Signal Processing*, 2000, submitted.

[4] T. Söderström and P. Stoica, *System Identification*, Prentice Hall International, London, UK, 1989.

[5] J. Li and P. Stoica, “An adaptive filtering approach to spectral estimation and SAR imaging,” *IEEE Transactions on Signal Processing*, vol. 44, no. 6, pp. 1469–1484, June 1996.

[6] H. Li, J. Li, and P. Stoica, “Performance analysis of forward-backward matched-filterbank spectral estimators,” *IEEE Transactions on Signal Processing*, vol. 46, no. 7, pp. 1954–1966, July 1998.

[7] J. Capon, “High resolution frequency-wavenumber spectrum analysis,” *Proceedings of the IEEE*, vol. 57, no. 8, pp. 1408–1418, August 1969.

[8] S. M. Kay, *Modern Spectral Estimation: Theory and Application*, Prentice Hall, Englewood Cliffs, NJ, 1988.

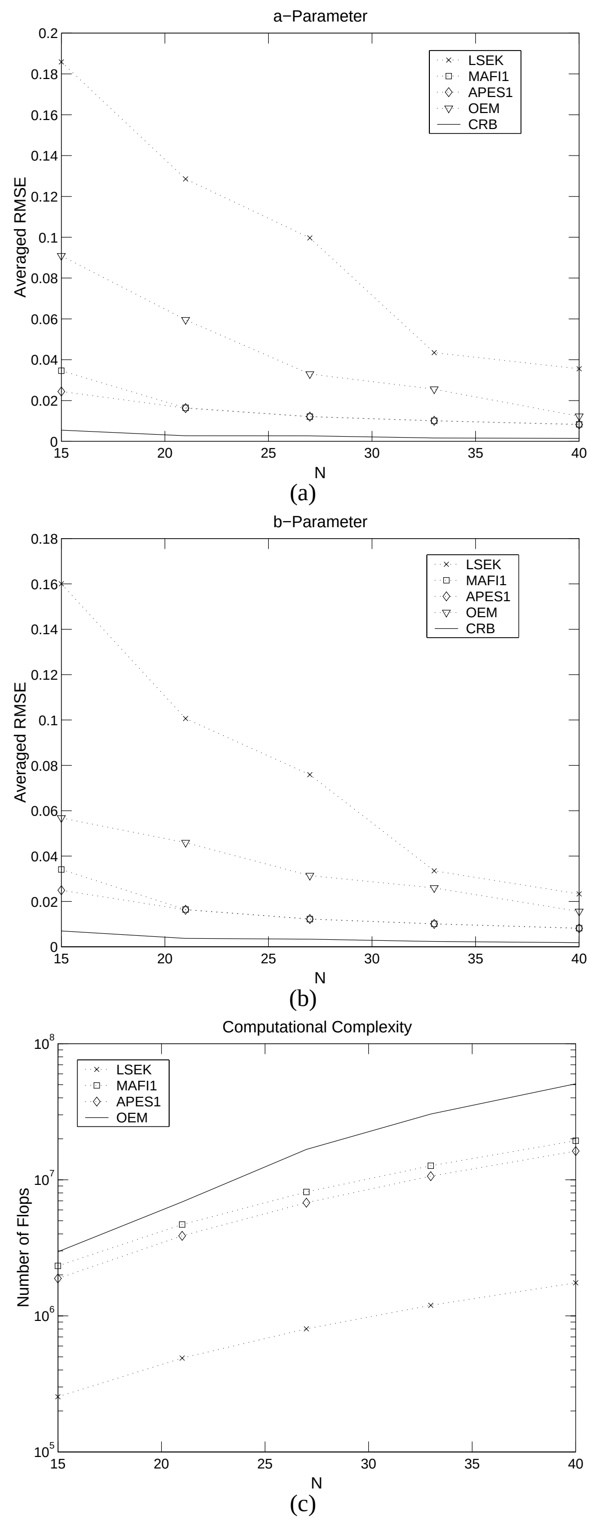


Fig. 2. Averaged RMSE and the number of flops vs. $N=\bar{N}$ with colored noise. (a) $\boldsymbol{a}$. (b) $\boldsymbol{b}$. (c) Number of flops.