ALMOST SURE IDENTIFIABILITY OF MULTIDIMENSIONAL HARMONIC RETRIEVAL

Tao Jiang[1], *Nicholas D. Sidiropoulos*[1*], *Jos M.F. ten Berge*[2]

[1] Department of Electrical and Computer Engineering, University of Minnesota
Minneapolis, MN 55455, USA; e-mail : (nikos,jiang)@ece.umn.edu
[2] Heijmans Instituut, Rijksuniversiteit Groningen
Grote Kruisstraat 2/1, 9712 TS Groningen, NL; e-mail : J.M.F.ten.BERGE@ppsw.rug.nl

ABSTRACT

Two-dimensional (2-D) and more generally multi-dimensional harmonic retrieval is of interest in a variety of applications. The associated identifiability problem is key in understanding the fundamental limitations of parametric high-resolution methods. In the 2-D case, existing identifiability results indicate that, assuming sampling at Nyquist or above, the number of resolvable exponentials is proportional to $I + J$, where I is the number of (equispaced) samples along one dimension, and J likewise for the other dimension. We prove in this paper that the number of resolvable exponentials is roughly $IJ/4$, almost surely. This is not far from the equations-versus-unknowns bound of $IJ/3$. We then generalize the result to the N-D case for any $N > 2$, showing that, under quite general conditions, the number of resolvable exponentials is proportional to total sample size, hence grows exponentially with the number of dimensions.

1. INTRODUCTION

The problem of harmonic retrieval is commonly encountered under different disguises in diverse applications in the sciences and engineering. Although one-dimensional harmonic retrieval is most common, many applications of multi-dimensional harmonic retrieval can be found in radar (e.g., [3, 6] and references therein), and wireless channel sounding (e.g., [2]), wherein one is interested in jointly estimating several multipath signal parameters like azimuth, elevation, delay, and Doppler, all of which can often be viewed as or transformed into frequency parameters.

A plethora of one-dimensional as well as multi-dimensional harmonic retrieval techniques have been developed, ranging from non-parametric Fourier-based methods, to modern parametric methods which are not bound by the Fourier resolution limit. In the high singal-to-noise ratio (SNR) regime, parametric methods work well with only a limited number of samples.

One important issue with parametric methods is to determine the maximum number of harmonics that can be resolved for a given total sample size; another is to determine the sample size needed to meet performance specifications.

Identifiability-imposed bounds on sample size are often not the issue in time series analysis, because samples are collected along the temporal dimension (hence "inexpensive"), and performance considerations dictate many more samples than what is needed for identifiability. The maximum number of resolvable harmonics comes back into play in situations where data samples along the harmonic mode come at a premium, e.g., in spatial sampling for direction-of-arrival estimation using a uniform linear array (ULA), in which case one can collect temporal samples to meet performance requirements.

Determining the maximum number of resolvable harmonics is a parameter identifiability problem, whose solution for the case of one-dimensional harmonics goes back to Carathéodory [1]; see also [8]. In two or higher dimensions, the identifiability problem is considerably harder, but also more interesting. The reason is that, in many applications of higher-dimensional harmonic retrieval, one is constrained in the number of samples that can be taken along certain dimensions, usually due to hardware and/or cost limitations. The question that arises is whether the number of samples taken along any particular dimension bounds the overall number of resolvable harmonics or not.

Essentially all of the work to date on identifiability of multi-dimensional harmonic retrieval deals with the 2-D case (e.g., [4, 6]), and provides sufficient identifiability conditions that are constrained by $\min(I, J)$, where I denotes the number of samples taken along one dimension, and J likewise for the other dimension. To the best of our knowledge, the most relaxed condition to date has been derived in [7], which shows that identifiability is determined by the *sum* $I + J$. The result of [7] is deterministic, in the sense that no statistical assumptions are needed aside from the requirement that the frequencies along *each* dimension are distinct. However, the sufficient condition in [7] improves with the *sum* of I, J, whereas total sample size grows with the *product* of I, J. This indicates that significantly stronger results are *possible.*

The focus of this paper is the derivation of stochastic identifiability results for 2-D harmonic retrieval, that fulfill this potential. Our tools allow us to treat the general case of multi-dimensional complex exponentials that incorporate real exponential components (e.g., decay rates). We thus make no distinction between the terms *harmonic* and *exponential.* We show that if the number of 2-D harmonics is less than or equal to roughly $IJ/4$, then, assuming sampling at the Nyquist rate or above, the parameterization (including the pairing of parameters) is $P_{\mathcal{L}}(\mathbb{C}^{2F})$-almost surely unique, where F is the number of harmonics and $P_{\mathcal{L}}(\mathbb{C}^{2F})$ is the distribution used to draw the $2F$ complex decay/frequency parameters, assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{2F}$. We then generalize this result to the N-D case.

The rest of paper is organized as follows. Section 2 establishes notation and preliminaries. Section 3 summarizes an earlier deterministic identifiability result, while section 4 develops some tools needed to prove our main stochastic identifiability result in section 5. This result is subsequently generalized to the N-D case in section 6. Conclusions are drawn in section 7.

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2. NOTATION AND SOME PRELIMINARIES

$\mathbb{C}$ denotes the complex numbers. Matrices (vectors) are denoted by boldface capital (lowercase) letters. We do not follow the usual convention of using i or j to denote $\sqrt{-1}$; instead we explicitly write $\sqrt{-1}$ when needed, and use i (j) as row (respectively, column) index, I (J) as row (respectively, column) size, in accordance with common practice in matrix algebra.

F denotes the number of harmonics, and $f \in \{1, \cdots, F\}$ is used as an index (note that f *is not* frequency in Hz). A sum of F 2-D exponentials can be written as

$$x_{i,j} = \sum_{f=1}^{F} c_f a_f^{i-1} b_f^{j-1},$$

where $c_f, a_f, b_f \in \mathbb{C}$.

The rank of a matrix (2-way array) $\mathbf{A}$ is the smallest number of rank-one matrices needed to decompose $\mathbf{A}$ into a sum of rank-one factors. Each rank-one factor is the outer product of two vectors. Matrix rank can be equivalently defined as the maximum number of linearly independent columns (or rows) that can be drawn from $\mathbf{A}$. We will use $r_{\mathbf{A}}$ to denote the rank of $\mathbf{A}$.

The Kruskal-rank or k-rank of a matrix $\mathbf{A}$ (denoted by $k_{\mathbf{A}}$) is r if *every* r columns of $\mathbf{A}$ are linearly independent, and either $\mathbf{A}$ has r columns or $\mathbf{A}$ contains a set of $r+1$ linearly dependent columns. The k-rank of $\mathbf{A}$ is therefore the maximum number of linearly independent columns that can be drawn from $\mathbf{A}$ in an *arbitrary fashion.* Note that k-rank is generically asymmetric: the k-rank of a matrix need not be equal to the k-rank of its transpose. k-rank is always less than or equal to rank.

An $m \times \rho$ Vandermonde matrix with generators $\alpha_1, \alpha_2, \cdots, \alpha_\rho \in \mathbb{C}$ is given by

$$\mathbf{V} := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_\rho \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_\rho^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{m-1} & \alpha_2^{m-1} & \cdots & \alpha_\rho^{m-1} \end{bmatrix}.$$

Let

$$\mathbf{A} = [\ \mathbf{a}_1 \ \cdots \ \mathbf{a}_i \ \cdots \ \mathbf{a}_r\ ],$$

$$\mathbf{B} = [\ \mathbf{b}_1 \ \cdots \ \mathbf{b}_i \ \cdots \ \mathbf{b}_r\ ],$$

be two matrices with common number of columns (r). The Khatri-Rao (column-wise Kronecker) matrix product of $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A} \odot \mathbf{B} := [\ \mathbf{a}_1 \otimes \mathbf{b}_1 \ \cdots \ \mathbf{a}_i \otimes \mathbf{b}_i \ \cdots \ \mathbf{a}_r \otimes \mathbf{b}_r\ ],$$

where $\mathbf{a}_i \otimes \mathbf{b}_i$ denotes the Kronecker product of $\mathbf{a}_i$ and $\mathbf{b}_i$.

3. DETERMINISTIC IDENTIFIABILITY

We will make use of the following result.

Theorem 1 *(Deterministic identifiability of N-dimensional harmonic retrieval [7]) Given a sum of F exponentials in N-dimensions*

$$x_{i_1,\cdots,i_N} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_n-1},$$

for $i_n = 1, \cdots, I_n \geq 2$, $n = 1, \cdots, N$, with $c_f \in \mathbb{C}$ and $a_{f,n} \in \mathbb{C}$ such that $a_{f_1,n} \neq a_{f_2,n}$, $\forall f_1 \neq f_2$ and all n, if

$$\sum_{n=1}^{N} I_n \geq 2F + (N-1),$$

then there exist unique $(a_{f,n},\ n = 1, \cdots, N;\ c_f)$, $f = 1, \cdots, F$ that give rise to $x_{i_1,\cdots,i_N}$. If an additional M non-exponential dimensions are available,

$$x_{i_1,\cdots,i_N,j_1,\cdots,j_M} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_n-1} \prod_{m=1}^{M} b_{f,m,j_m},$$

for $j_m = 1, \cdots, J_m \geq 2$, $m = 1, \cdots, M$, with $b_{f,m,1} = 1$, $\forall f, m$ by convention, then uniqueness (including the associated component vectors along non-exponential dimensions) holds provided that

$$\sum_{n=1}^{N} I_n + \sum_{m=1}^{M} k_{\mathbf{B}^{(m)}} \geq 2F + (N+M-1),$$

where $\mathbf{B}^{(m)}$ denotes the $J_m \times F$ matrix with (j_m, f) element b_{f,m,j_m}.

4. ON RANK AND K-RANK OF THE KHATRI-RAO PRODUCT

In order to improve Theorem 1, we need to find a sufficient condition under which the Khatri-Rao product of two Vandermonde matrices is full k-rank. We can show that, given any Vandermonde matrix $\mathbf{A}$, we can always find another Vandermonde matrix $\mathbf{B}$ such that the Khatri-Rao product $\mathbf{A} \odot \mathbf{B}$ is rank-deficient. Therefore, we cannot expect to find separable deterministic conditions on the generators of $\mathbf{A}$, $\mathbf{B}$ to guarantee that $\mathbf{A} \odot \mathbf{B}$ has full rank. Other researchers have noted that the Khatri-Rao product appears to exhibit full rank in essentially all cases of practical interest [9], but no rigorous argument has been given to justify this observation to date. The following two results settle this issue.

Theorem 2 *For a pair of Vandermonde matrices $\mathbf{A} \in \mathbb{C}^{I \times F}$ and $\mathbf{B} \in \mathbb{C}^{J \times F}$*

$$r_{\mathbf{A}\odot\mathbf{B}} = k_{\mathbf{A}\odot\mathbf{B}} = min(IJ, F), \quad P_{\mathcal{L}}(\mathbb{C}^{2F}) - a.s., \qquad (1)$$

where $P_{\mathcal{L}}(\mathbb{C}^{2F})$ is the distribution used to draw the $2F$ complex generators for $\mathbf{A}$ and $\mathbf{B}$, assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{2F}$.

Proof: The general case can be reduced to the $IJ = F$ case. If $IJ \leq F$, it suffices to prove that the result holds for an *arbitrary* selection of IJ columns; if $IJ \geq F$, then it suffices to prove that the result holds for any row-reduced square sub-matrix. When $IJ = F$, full rank and full k-rank can be established by showing that the determinant of $\mathbf{A} \odot \mathbf{B}$ is nonzero. Define

$$H(\alpha_1, \cdots, \beta_F) = det(\mathbf{A}(\alpha_1, \cdots, \alpha_F) \odot \mathbf{B}(\beta_1, \cdots, \beta_F)).$$

H is a polynomial in $2F$ variables, hence analytic. In order to establish the desired result, it suffices to show that H is non-trivial. This requires a "generic" example, that works for any I, J, F. This can be constructed as follows. For any given I, J, F with $2 \leq I \leq F$ and $2 \leq J \leq F$, $IJ = F$, define the generators $\alpha_f = e^{\sqrt{-1}\frac{2\pi}{F}J(f-1)}$, and $\beta_f = e^{\sqrt{-1}\frac{2\pi}{F}(f-1)}$ for $f = 1, \cdots, F$. It can be verified that, with this choice of generators for $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \odot \mathbf{B}$ is itself a Vandermonde matrix with generators $(1, e^{\sqrt{-1}\frac{2\pi}{F}}, \cdots, e^{\sqrt{-1}\frac{2\pi}{F}(F-1)})$, and therefore full rank. This shows that $H(\alpha_1, \cdots, \alpha_F, \beta_1, \cdots, \beta_F)$ is a non-trivial polynomial in $\mathbb{C}^{2F}$, hence a non-trivial analytic function in $\mathbb{C}^{2F}$. By the fact that the zero set of a non-trivial analytic function has zero Lebesgue measure (e.g., [8](p.268), and also [5] for a simple proof), $H(\alpha_1, \cdots, \alpha_F, \beta_1, \cdots, \beta_F)$ is non-zero almost everywhere, except for a measure zero subset of $\mathbb{C}^{2F}$. $\square$

As an almost direct by-product, we obtain:

Corollary 1 *For a pair of matrices $\mathbf{A} \in \mathbb{C}^{I \times F}$ and $\mathbf{B} \in \mathbb{C}^{J \times F}$,*

$$r_{\mathbf{A}\odot\mathbf{B}} = k_{\mathbf{A}\odot\mathbf{B}} = min(IJ, F), \quad P_{\mathcal{L}}(\mathbb{C}^{(I+J)F}) - a.s., \qquad (2)$$

where $P_{\mathcal{L}}(\mathbb{C}^{(I+J)F})$ is the distribution used to draw the $(I+J)F$ complex elements of A and B, assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{(I+J)F}$.

The proof of Corollary 1 can be found in [5]. Equipped with these results, we proceed to address the main problem of interest herein.

5. ALMOST SURE IDENTIFIABILITY OF 2-D HARMONIC RETRIEVAL

Proposition 1 [1] *Given a sum of F 2-D exponentials*

$$x_{i,j} = \sum_{f=1}^{F} c_f a_f^{i-1} b_f^{j-1}, \qquad (3)$$

for $i = 1, \ldots, I \geq 4$, and $j = 1, \ldots, J \geq 4$, the parameter triples (a_f, b_f, c_f), $f = 1, \cdots, F$ are $P_{\mathcal{L}}(\mathbb{C}^{2F})$-a.s. unique[2]*, where $P_{\mathcal{L}}(\mathbb{C}^{2F})$ is the distribution used to draw the $2F$ complex exponential parameters (a_f, b_f), $f = 1, \cdots, F$, assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{2F}$, provided that there exist four integers, I_1, I_2, J_1, J_2 such that*

$$I - I_1 - I_2 + min(I_1 J_1, F) + min(I_2 J_2, F) \geq 2F, \qquad (4)$$

subject to

$$I_1 + I_2 \leq I, \quad J_1 + J_2 = J + 1, \quad \min(I_1, I_2, J_1, J_2) \geq 2. \qquad (5)$$

Proof: We first define a 5-way array with typical element

$$\begin{aligned} \widehat{x}_{i_1,i_2,i_3,j_1,j_2} &:= x_{i_1+i_2+i_3-2, j_1+j_2-1} \\ &= \sum_{f=1}^{F} c_f a_f^{i_1+i_2+i_3-1-1-1} b_f^{j_1+j_2-1-1} \\ &= \sum_{f=1}^{F} c_f a_f^{i_1-1} a_f^{i_2-1} a_f^{i_3-1} b_f^{j_1-1} b_f^{j_2-1} \end{aligned}$$

where $i_\alpha = 1, \cdots, I_\alpha \geq 2$, and $j_\beta = 1, \cdots, J_\beta \geq 2$, for $\alpha = 1, 2, 3$, $\beta = 1, 2$. Since $\min(I, J) \geq 4$ has been assumed in the statement of the proposition, such extension to 5 ways is always feasible. Define matrices

$$\mathbf{A}_\alpha = (a_f^{i_\alpha - 1}) \in \mathbb{C}^{I_\alpha \times F}, \quad \mathbf{B}_\beta = (b_f^{j_\beta - 1}) \in \mathbb{C}^{J_\beta \times F}.$$

The next step is to nest the 5-way array $\widehat{x}$ into a three-way array $\bar{x}$ by collapsing two pairs of dimensions as follows

$$\begin{aligned} \bar{x}_{i_3,k,l} &:= \widehat{x}_{\lceil \frac{k}{J_1} \rceil, \lceil \frac{l}{J_2} \rceil, i_3, k-(\lceil \frac{k}{J_1} \rceil - 1)J_1, l-(\lceil \frac{l}{J_2} \rceil -1)J_2} \\ &= \sum_{f=1}^{F} c_f a_f^{\lceil \frac{k}{J_1} \rceil - 1} a_f^{\lceil \frac{l}{J_2} \rceil - 1} a_f^{i_3-1} \times \\ &\qquad \times b_f^{k-(\lceil \frac{k}{J_1} \rceil - 1)J_1 - 1} b_f^{l-(\lceil \frac{l}{J_2} \rceil - 1)J_2 - 1} \\ &= \sum_{f=1}^{F} c_f a_f^{i_3-1} a_f^{\lceil \frac{k}{J_1} \rceil - 1} b_f^{k-(\lceil \frac{k}{J_1} \rceil - 1)J_1 - 1} \times \\ &\qquad \times a_f^{\lceil \frac{l}{J_2} \rceil - 1} b_f^{l-(\lceil \frac{l}{J_2} \rceil - 1)J_2 - 1} \\ &= \sum_{f=1}^{F} c_f a_f^{i_3-1} d_{k,f} e_{l,f}, \end{aligned}$$

for $k = 1, \cdots, I_1 J_1$, $l = 1, \cdots, I_2 J_2$, with $d_{k,f}$ and $e_{l,f}$ given by

$$d_{k,f} := a_f^{\lceil \frac{k}{J_1} \rceil - 1} b_f^{k-(\lceil \frac{k}{J_1} \rceil - 1)J_1 - 1}, e_{l,f} := a_f^{\lceil \frac{l}{J_2} \rceil - 1} b_f^{l-(\lceil \frac{l}{J_2} \rceil - 1)J_2 - 1}$$

Define matrices

$$\mathbf{D} = (d_{k,f}) \in \mathbb{C}^{I_1 J_1 \times F}, \quad \mathbf{E} = (e_{l,f}) \in \mathbb{C}^{I_2 J_2 \times F}$$

$\mathbf{D}$ and $\mathbf{E}$ are nothing but

$$\mathbf{D} = \mathbf{A}_1 \odot \mathbf{B}_1, \quad \mathbf{E} = \mathbf{A}_2 \odot \mathbf{B}_2$$

Since $\mathbf{A}_3$ is Vandermonde, Theorem 1 can be invoked to claim uniqueness, provided

$$I_3 + k_{\mathbf{D}} + k_{\mathbf{E}} \geq 2F + 3 - 1. \qquad (6)$$

Note that for any particular i_3, k and l, the product $c_f a_f^{i_3-1} d_{k,f} e_{l,f}$ is equal to $c_f a_f^{i-1} b_f^{j-1}$ with the following choice of i and j:

$$i = i_3 + \lceil \frac{k}{J_1} \rceil + \lceil \frac{l}{J_2} \rceil - 2,$$

$$j = k - (\lceil \frac{k}{J_1} \rceil - 1)J_1 + l - (\lceil \frac{l}{J_2} \rceil - 1)J_2 - 1.$$

As i_3, k and l span their range, the corresponding i and j span their respective range. It follows that uniqueness of the F rank-one 3-D factors $c_f a_f^{i_3-1} d_{k,f} e_{l,f}$ is equivalent to uniqueness of the F rank-one 2-D factors $c_f a_f^{i-1} b_f^{j-1}$, $f = 1, \cdots, F$. Therefore, the rank-one factors $c_f a_f^{i-1} b_f^{j-1}$ and hence the triples (a_f, b_f, c_f), $f = 1, \cdots, F$, are unique provided that (6) holds true. Invoking Theorem 2, almost sure uniqueness holds provided there exist integers, $I_1, I_2, I_3, J_1, J_2 \geq 2$ such that

$$I_3 + min(I_1 J_1, F) + min(I_2 J_2, F) \geq 2F + 2,$$

[1] The result holds true if we switch I and J.

[2] We assume throughout that sampling is at the Nyquist rate or higher, to avoid spectral folding. This allows us to restrict attention to discrete-time frequencies in $(-\pi, \pi]$.

subject to[3]

$$\begin{aligned} I_1 + I_2 + I_3 = I + 2, \; J_1 + J_2 = J + 1, \\ \min(I_1, I_2, I_3, J_1, J_2) \geq 2. \end{aligned}$$

Setting $I_3 = I + 2 - I_1 - I_2$, we obtain

$$I - I_1 - I_2 + min(I_1 J_1, F) + min(I_2 J_2, F) \geq 2F,$$

subject to

$$I_1 + I_2 \leq I, \; J_1 + J_2 = J + 1, \; \min(I_1, I_2, J_1, J_2) \geq 2,$$

and the proof is complete. □

5.1. Main 2-D result

Theorem 3 [4] *Given a sum of F 2-D exponentials*

$$x_{i,j} = \sum_{f=1}^{F} c_f a_f^{i-1} b_f^{j-1}, \tag{7}$$

for $i = 1, \ldots, I \geq 4$, and $j = 1, \ldots, J \geq 4$, the parameter triples (a_f, b_f, c_f), $f = 1, \cdots, F$ are $P_{\mathcal{L}}(\mathbb{C}^{2F})$-a.s. unique, where $P_{\mathcal{L}}(\mathbb{C}^{2F})$ is the distribution used to draw the $2F$ complex exponential parameters (a_f, b_f), $f = 1, \cdots, F$, assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{2F}$, provided that

$$F \leq \lfloor \frac{I}{2} \rfloor \lceil \frac{J}{2} \rceil \tag{8}$$

Proof: If I is even, pick $I_1 = I_2 = \frac{I}{2}$, otherwise, pick $I_1 = \frac{I-1}{2}$ and $I_2 = \frac{I+1}{2}$ (thereby satisfying condition (5)). If J is even, pick $J_1 = \frac{J}{2}$ and $J_2 = \frac{J+2}{2}$, otherwise, let $J_1 = J_2 = \frac{J+1}{2}$ (hence satisfying condition (5)).
Once we pick four integers following the above rules, condition (8) assures that inequality (4) holds for those particular integers. Invoking Proposition 1 completes the proof. □

Remark 1 *It is interesting to note that equations-versus-unknowns considerations indicate a bound of $IJ/3$, without taking the pairing issue into consideration. To see this, note that each of the F 2-D exponential components is parameterized by 3 complex parameters, and a total of IJ complex data points are given. If the equations-versus-unknowns bound is violated, then the implicit function theorem indicates that infinitely many ambiguous solutions exist in the neighborhood of the true solution.*

6. ALMOST SURE IDENTIFIABILITY OF N-D HARMONIC RETRIEVAL

We now state our result for the N-D case. The proof is omitted due to space limitations, but is included in the associated journal paper [5].

Theorem 4 [5] *Given a sum of F N-D exponentials*

$$x_{i_1,\cdots,i_N} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_n - 1}, \tag{9}$$

for $i_n = 1, \cdots, I_n \geq 4$, $n = 1, \cdots, N$, the parameter $(N+1)$-tuples $(a_{f,1}, \cdots, a_{f,N}, c_f)$, $f = 1, \cdots, F$, are $P_{\mathcal{L}}(\mathbb{C}^{NF})$-a.s. unique, where $P_{\mathcal{L}}(\mathbb{C}^{NF})$ is the distribution used to draw the NF complex exponential parameters $(a_{f,1}, \cdots, a_{f,N})$, for $f = 1, \cdots, F$, assumed continuous with respect to Lebesgue measure in $\mathbb{C}^{NF}$, provided that

$$F \leq \lfloor \frac{I_1}{2} \rfloor \prod_{n=2}^{N} \lceil \frac{I_n}{2} \rceil \tag{10}$$

7. CONCLUSIONS

We have derived stochastic identifiability results for 2-D and N-D harmonic retrieval. The associated identifiability conditions are the most relaxed to date. The sufficient condition for the 2-D case is not far from equations-versus-unknowns considerations - hence additional improvements, if any, will be marginal. In the N-D case, the bound is still proportional to total sample size - hence grows exponentially in the number of dimensions - but deviates from the equations-versus-unknowns bound. This indicates that the sufficient condition provided herein could be improved in higher dimensions.

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[3] The first two conditions assure that we do not index beyond the available data sample.

[4] The Theorem holds true if I and J are switched.

[5] The Theorem holds true for any permutation of $\{I_n\}_{n=1}^{N}$