A ROOT-MUSIC ALGORITHM FOR NON CIRCULAR SOURCES

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ABSTRACT

We present in this paper a new direction finding algorithm for non circular sources that is based on polynomial rooting. Due to the non circularity characteristics of the impinging sources, the proposed method is able to handle more sources than sensors. By using a polynomial rooting instead of a searching technique the method is limited to linear uniformly spaced arrays. However, polynomial rooting reduces significantly computation cost and enhances resolution power. Computer simulations are used to show the performance of the algorithm.

1. INTRODUCTION

The problem of estimating the direction-of-arrival (DOA) of narrowband sources from sensor array data has received considerable attention. This estimation can be obtained from an array of antennas mounted on vehicles, ships, aircrafts, satellites, and base stations. The satellite communications systems of the future generation also require the knowledge of DOA to supply different functions, such as sorting and routing of communications towards remote mobiles, handover management, or geographic localization of terminals. In [1], beamforming and DOA estimation methods have been suggested for mobile communications systems. Due to the today's rapidly growing wireless mobile communications market, the mobile network operators have to increase the capacity of their network. Nowadays, we need systems capable of receiving and separating more and more impinging signals.

Moreover, many studies about complex random variables and signals have been led, in which a great interest for the non circular signals has been shown as in [2] and [3]. Actually many systems now deal with non circular incoming signals, as in telecommunication or satellite systems where amplitude modulated (AM) or binary phase-shift keying (BPSK) modulated signals are often used.
Galy proposed in [4] a MUSIC-like algorithm (called NC-MUSIC) dealing with such modulated signals. We present in this paper a Root-MUSIC-like (Root-MUSIC [5]) direction finding algorithm designed for non circular signals. The proposed algorithm is restricted to linear uniformly spaced arrays, but has the distinct advantage over the multiple signal classification (MUSIC) algorithms [6] and [4] that it does not require a search over parameter space. Instead, our algorithm requires finding the roots of a polynomial, which is very simple and has a low computation cost.

In Section 2, the problem is formulated. By giving the model of signals and a description of the general characteristics of a non circular random variable, the array data model extended to the case of non circular emitting sources is then provided. In Section 3, the proposed direction finding procedure is presented. Section 4 includes simulation results that show the performance of the proposed method. Finally, Section 5 contains our conclusions.

2. PROBLEM FORMULATION

In this paper we consider a uniform linear array of m sensors. The distance between two adjacent antennas is noted δ. Suppose d electromagnetic waves impinging on the array from angular directions θ_k, $k = 1, \ldots, d$. The incident waves are assumed to be plane waves, as generated from far-field point sources. Furthermore, the signals are assumed to be narrowband.

2.1. Array data model

The global signal received by the array from the emitting narrowband sources is assumed to obey the following model:

$$\mathbf{z}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \qquad (1)$$

where the vector $\mathbf{s}(t) = [s_1(t), \ldots, s_d(t)]^T$ contains temporal signals transmitted by the d sources, the vector $\mathbf{n}(t)$ represents spatially white noise. $(.)^T$ denotes the transpose operator. The matrix $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_d)]$ contains the steering vectors of the impinging sources. The steering vector of the k^{th} source is then:

$$\mathbf{a}(\theta_k) = [1, e^{j\frac{2\pi\delta}{\lambda}\sin(\theta_k)}, \ldots, e^{j\frac{2\pi\delta}{\lambda}(m-1)\sin(\theta_k)}]^T \qquad (2)$$

where λ is the wavelength of the impinging signal. Assuming that noise and signals are uncorrelated and that noise is spatially white, data model (1) allows us to write the covariance matrix of the array measurements as:

$$\mathbf{R} = E[\mathbf{z}(t)\mathbf{z}^H(t)] = \mathbf{R_S} + \sigma^2\mathbf{I} \tag{3}$$

with

$$\mathbf{R_S} = \mathbf{A}\Gamma_\mathbf{S}\mathbf{A}^H \tag{4}$$

where

$$\Gamma_\mathbf{S} = E[\mathbf{s}(t)\mathbf{s}^H(t)] \tag{5}$$

is the emitted signal covariance matrix and $\sigma^2\mathbf{I}$ is the noise covariance matrix. $E[.]$ denotes expectation and $(.)^H$ denotes the complex conjugate transpose operator.

2.2. Non circular signals

Circularity is an important property of random variables which is depicted in [2] and [3]. The concept of circularity directly comes from the geometrical interpretation of complex random variables.

Here we use only the first and the second orders statistical properties of the signals. The definition of circularity for order 2 is very simple. For a complex random variable, x, the only moments to be considered are the mean $E[x]$, the covariance $E[xx^*]$, and the elliptic covariance $E[xx]$. A complex random variable is said to be circular at the order 2, if both the mean and the elliptic covariance equal zero. The second order statistical characteristics of x are so contained in its covariance $E[xx^*]$. Circularity is a common hypothesis for narrowband signals analysis, but we can easily find numerous non circular signals, like AM or BPSK modulated signals.

In our study, we assume that sources emit non circular signals. Using this assumption, we can give as in [4] an extension of the classical model (1). By concatenating the array measurements and their conjugate components, the following observation vector can be written:

$$\mathbf{z}_{nc}(t) = \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{z}^*(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^* \end{bmatrix} \begin{bmatrix} \mathbf{s}(t) \\ \mathbf{s}^*(t) \end{bmatrix} + \begin{bmatrix} \mathbf{n}(t) \\ \mathbf{n}^*(t) \end{bmatrix} \tag{6}$$

where $(.)^*$ denotes the complex conjugate operator. Then, when sources emit AM or BPSK modulated signals, we can form the extended covariance matrix of the observations as:

$$\begin{aligned} \mathbf{R}_{nc} &= E[\mathbf{z}_{nc}(t)\mathbf{z}_{nc}^H(t)] \\ &= \begin{bmatrix} \mathbf{A} \\ \mathbf{A}^*\Psi^* \end{bmatrix} \Gamma_\mathbf{S} \begin{bmatrix} \mathbf{A} \\ \mathbf{A}^*\Psi^* \end{bmatrix}^H + \sigma^2\mathbf{I} \end{aligned} \tag{7}$$

where Ψ is a diagonal matrix, and each diagonal element is a natural phase $e^{j\psi_k}$ relative to an impinging source. By eigendecomposition, a d-dimensional signal subspace and an orthogonal $(2m-d)$-dimensional subspace can be determined. This extended model allows to increase the observation space while keeping the dimension of the signal subspace unchanged. The extended steering vector is then:

$$\mathbf{b}(\theta, \psi) = \begin{bmatrix} \mathbf{a}(\theta) \\ \mathbf{a}^*(\theta)e^{-j\psi} \end{bmatrix} \tag{8}$$

3. DOA ESTIMATION

The extended covariance matrix (7) takes the following form $\mathbf{R}_{nc} = \mathbf{U}\Sigma\mathbf{U}^H$ by eigendecomposition. Assuming that the $2m$ diagonal elements of Σ have been arranged in decreasing order, the well-ordered eigenvalues are used to determine d, the number of sources. Once d is known, under the assumption that the sources are non circular, we then estimate the DOA.

According to the orthogonality between eigenvectors associated with the $2m-d$ smallest eigenvalues of $\mathbf{R}_{nc}$ and the theoretical steering vectors, DOA estimates are obtained by minimizing the following cost function:

$$J(\theta, \psi) = \mathbf{b}^H(\theta, \psi)\mathbf{U}_n\mathbf{U}_n^H\mathbf{b}(\theta, \psi) \tag{9}$$

where the $2m-d$ eigenvectors of $\mathbf{R}_{nc}$ spanning the signal nullspace form the matrix $\mathbf{U}_n$. The cost function can be written as:

$$J(\theta, \psi) = \mathbf{q}^H\mathbf{M}\mathbf{q} \tag{10}$$

where

$$\mathbf{q} = \begin{bmatrix} 1 \\ e^{-j\psi} \end{bmatrix} \tag{11}$$

and $\mathbf{M}$ is a (2×2) matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{a}^H(\theta)\mathbf{U}_{n1}\mathbf{U}_{n1}^H\mathbf{a}(\theta) & \mathbf{a}^H(\theta)\mathbf{U}_{n1}\mathbf{U}_{n2}^H\mathbf{a}^*(\theta) \\ \mathbf{a}^T(\theta)\mathbf{U}_{n2}\mathbf{U}_{n1}^H\mathbf{a}(\theta) & \mathbf{a}^T(\theta)\mathbf{U}_{n2}\mathbf{U}_{n2}^H\mathbf{a}^*(\theta) \end{bmatrix} \tag{12}$$

where $\mathbf{U}_{n1}$ and $\mathbf{U}_{n2}$ are two submatrices of the same dimension and:

$$\mathbf{U}_n = \begin{bmatrix} \mathbf{U}_{n1} \\ \mathbf{U}_{n2} \end{bmatrix} \tag{13}$$

It can be shown [4] that $\mathbf{U}_{n2}^*\mathbf{U}_{n2}^T = \mathbf{U}_{n1}\mathbf{U}_{n1}^H$. Thus, the two diagonal elements of the matrix $\mathbf{M}$ are equal. Note also that the two non diagonal elements of $\mathbf{M}$ form a reciprocal conjugate pair.

The minimum of the quadratic form in (10) over θ and ψ is given by the smallest eigenvalue of the matrix $\mathbf{M}$. This eigenvalue is always nonnegative since the quadratic form is nonnegative. When θ is a true DOA, the smallest eigenvalue of $\mathbf{M}$ is equal to zero, and then the determinant of the matrix $\mathbf{M}$ equals zero too.

Let's now define the complex variable z,

$$z = e^{j\frac{2\pi\delta}{\lambda}\sin(\theta)} \tag{14}$$

so that $\mathbf{a}(\theta)$ can be written as

$$\mathbf{a}(z) = \left[1, z, z^2, \ldots, z^{m-1}\right]^T \tag{15}$$

Then the matrix $\mathbf{M}$ is a function of z. We estimate the DOAs by finding the values of z such that

$$\det\{\mathbf{M}\} = 0 \tag{16}$$

The left side of (16) is a polynomial of z. The DOA estimation problem is then transformed into a polynomial root-finding problem that can be solved using computationally efficient root-solving algorithms.

The polynomial of z can take the following form:

$$\det\{\mathbf{M}\} = m_1^2 - m_2\, m_3 \tag{17}$$

where

$$\begin{cases} m_1 &= \mathbf{a}^T(1/z)\mathbf{U}_{n1}\mathbf{U}_{n1}^H\mathbf{a}(z) \\ m_2 &= \mathbf{a}^T(1/z)\left(\mathbf{U}_{n2}\mathbf{U}_{n1}^H\right)^H\mathbf{a}(1/z) \\ m_3 &= \mathbf{a}^T(z)\mathbf{U}_{n2}\mathbf{U}_{n1}^H\mathbf{a}(z) \end{cases} \tag{18}$$

Therefore m_1 is a polynomial in z whose l^{th} coefficient is given by the sum of the elements of the l^{th} diagonal of $\mathbf{U}_{n1}\mathbf{U}_{n1}^H$, where $l = -m+1$ indicates the lowest diagonal and $l = m-1$ indicates the highest diagonal. Let $\mathbf{c} = [c_1, \ldots, c_{2m-1}]^T$ be the column vector of the coefficients of the polynomial m_1. Hence

$$m_1 = \sum_{i=1}^{2m-1} c_i z^{i-m} \tag{19}$$

The coefficients of the polynomial m_1^2 equal the sums of the antidiagonal elements of the matrix $\mathbf{c}\mathbf{c}^T$. Let $\mathbf{s} = [s_1, \ldots, s_{4m-3}]^T$ be the vector containing these $4m-3$ coefficients. We obtain

$$m_1^2 = \sum_{i=1}^{4m-3} s_i z^{i-(2m-1)} \tag{20}$$

The matrix $\mathbf{U}_{n1}\mathbf{U}_{n1}^H$ being an hermitian matrix, the elements of the vector $\mathbf{c}$ have the symmetry property $c_i = c_{2m-i}^*$. Since $\mathbf{c}\mathbf{c}^T$ is a symmetrical matrix, the coefficients of the polynomial m_1^2 keep the same property of symmetry $s_i = s_{4m-2-i}^*$.

In the same way, $\mathbf{u}$ is a column vector containing the sum of the $2m-1$ antidiagonal elements of the matrix $\mathbf{U}_{n2}\mathbf{U}_{n1}^H$. Then

$$m_2 = \sum_{i=1}^{2m-1} u_i^* z^{-(i-1)} \tag{21}$$

$$m_3 = \sum_{i=1}^{2m-1} u_i z^{i-1} \tag{22}$$

and $\mathbf{p}$ is the column vector whose elements are the sum of the diagonal elements of the matrix $\mathbf{u}^*\mathbf{u}^T$. Hence

$$m_2\, m_3 = \sum_{i=1}^{4m-3} p_i z^{i-(2m-1)} \tag{23}$$

The matrix $\mathbf{u}^*\mathbf{u}^T$ being an hermitian matrix, the coefficients of the polynomial $m_2\, m_3$ also have the property of symmetry $p_i = p_{4m-2-i}^*$.

Equation (16) can now be written:

$$\det\{\mathbf{M}\} = \sum_{i=1}^{4m-3} \left(s_i - p_i\right) z^{i-(2m-1)} = 0 \tag{24}$$

The roots of the polynomial $\det\{\mathbf{M}\}$ can be computed using any polynomial root-finding algorithm. The DOA estimates are obtained using (14):

$$\theta_k = \arcsin\left(\frac{\lambda}{2\pi\delta}\arg(z_n)\right) \tag{25}$$

where z_n represents one of the d roots selected for DOA estimation. Due to the symmetry property of the polynomial coefficients, roots appear in reciprocal conjugate pairs z_i and $1/z_i^*$. In each pair one root is inside the unit circle while the other is outside the unit circle (the two roots coincide if they are on the unit circle). Either one of the two can be used for DOA estimation, since they have the same angle in the complex plane. We can decide to use the roots inside the unit circle. We then select the d roots that are nearest to the unit circle as being the roots corresponding to the DOA estimates.

Note that the degree of the polynomial $\det\{\mathbf{M}\}$ is $4m-4$ ($4m-3$ coefficients). Hence the number of roots is $4m-4$, and since roots appear in reciprocal pairs, the proposed procedure allows to determine until $2(m-1)$ possible DOA. This has to be emphasized since the number of DOA estimates can be larger than the number of sensors. This characteristic is due to the extended data model provided by the non circularity property of the sources.

It is also of interest to estimate the natural phase ψ_k of an impinging signal, as well as the DOA. Recalling that the minimum of the quadratic form in (10) is obtained when this form is equal to the smallest eigenvalue of $\mathbf{M}$, then ψ_k is given by the eigenvector $\mathbf{q}$ associated to this eigenvalue. The smallest eigenvalue being equal to zero, and the corresponding DOA θ_k being known, we then obtain easily the expression of the associated eigenvector $\mathbf{q}$:

$$\mathbf{q} = \begin{bmatrix} 1 \\ -\frac{\mathbf{a}^T(\theta_k)\mathbf{U}_{n2}\mathbf{U}_{n1}^H\mathbf{a}(\theta_k)}{\mathbf{a}^H(\theta_k)\mathbf{U}_{n1}\mathbf{U}_{n1}^H\mathbf{a}(\theta_k)} \end{bmatrix} \tag{26}$$

Comparing equations (11) and (26), we determine the natural phase corresponding to the DOA θ_k as:

$$\psi_k = \pi - \arg\left(\mathbf{a}^T(\theta_k)\mathbf{U}_{n2}\mathbf{U}_{n1}^H\mathbf{a}(\theta_k)\right) \tag{27}$$

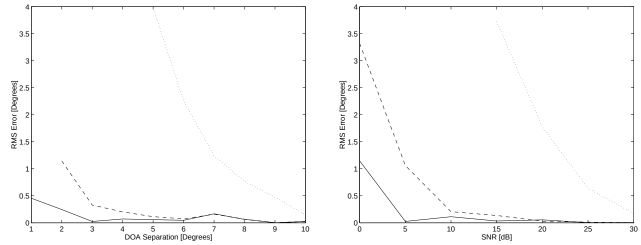
4. SIMULATION RESULTS

In this section we present some simulation results that illustrate the performance of the proposed algorithm. We also compare simulation results of the proposed procedure with those of the NC-MUSIC method [4] and the classical MUSIC algorithm [6]. Consider a uniform linear array of 6 sensors separated by a half wavelength of the incoming narrowband signals. The performance of the estimators in each of the simulations below is obtained from 100 Monte-Carlo simulations, by calculating the root-mean-square error (RMS Error) of DOA estimates. Sources in presence emit BPSK modulated signals with rectangular pulse shape. The number of data samples taken at each sensor output is 200.

Figure1(a) shows how the DOA separation affects the performance of the estimators. Two uncorrelated BPSK signals are generated with signal-to-noise ratio (SNR) of 10 dB. The angular separation between the sources is varied from $1°$ to $10°$. As expected the DOA estimation accuracy improves with increasing angular separation. When sources become close, the classical MUSIC fails in separating the two incoming signals, and the proposed algorithm performs better than the two others. For well separated sources, there is no difference between the performance of our polynomial root-finding technique and NC-MUSIC.

Figure1(b) shows behaviours of the estimators when SNR is varied from 0 dB to 30 dB. Two equal power uncorrelated BPSK signals $3°$ apart impinge on the linear uniformly spaced array of 6 sensors. Performances obtained with procedures taking non circularity of the sources into account are always better. Since the NC-MUSIC algorithm and the proposed polynomial root-finding algorithm use the same extended data model, they perform identically in easy case (high SNR). However for low SNR the proposed procedure provides more accurate DOA estimates.

Finally, when the classical MUSIC or the NC-MUSIC algorithm is used to estimate DOA, one must search over the set of all possible arrival angles to obtain the estimates. The proposed procedure avoids the problem of search entirely, and the computation cost is then much lower.



(a) (b)

Fig. 1. (a) RMS Error versus DOA Separation and (b) RMS Error versus SNR for : (solid line) proposed procedure, (dashed line) NC-MUSIC and (dotted line) Classical MUSIC.

5. CONCLUSION

In this paper we have described a computationally efficient procedure for DOA estimation. By assuming that incoming signals are AM or BPSK modulated signals, the algorithm uses the non circularity property of the signals to improve the estimations performance. Moreover by using a polynomial root-finding, the proposed algorithm does not require an explicit search procedure, and hence reduces considerably the computational requirements.

The performance of the proposed algorithm was evaluated by computer simulations, and was compared with that obtained by two other techniques. In these numerical comparisons, it can be seen that the proposed method exhibits better estimation performance. We have then verified the expected benefits due to the non circularity property.

The current approach may also be extended to deal with other kinds of signals. A study is in progress in order to exploit this issue.

6. REFERENCES

[1] L. C. Godara, “Application of antenna arrays to mobile communications, Part II: beam-forming and direction-of-arrival considerations,” Proceedings of the IEEE, vol. 85, pp. 1193-1245, August 1997.

[2] B. Picinbono, “On circularity,” IEEE Transactions on Signal Processing. vol. 42, pp. 3473-3482, December 1994.

[3] J.L. Lacoume, “Complex random variables and signals,” Traitement du Signal. vol. 15, pp. 535-544, 1998.

[4] J. Galy, “Antenne adaptative: du second ordre aux ordres supérieurs, applications aux signaux de télécommunications,” Toulouse, Ph. Dissertation,1998.

[5] A. J. Barabell, “Improving the resolution performance of eigenstructure based direction finding algorithms,” Proceedings IEEE International Conference on Acoustic, Speech, Signal Processing. 1983, pp.336-339.

[6] R. O. Schmit, “Multiple emitter location and signal parameters estimation,” IEEE Transactions on Antennas and Propagation. vol. 34, pp. 276-280, March 1986.