A Quantitative SNR Analysis of Linear Chirps in the Continuous-Time Short-Time Fourier Transform Domain with Gaussian Windows

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Abstract

In this paper, we present a quantitative signal-to-noise ratio (SNR) analysis of linear chirps in the continuous-time short-time Fourier transform (STFT) domain using Gaussian windows, the 3dB SNR definition is used. It is compared with the SNRs in the time and the Fourier transform domains. Some numerical examples are shown to illustrate the theory.

1 Introduction

It is known that a joint time-frequency analysis (JTFA) concentrates a chirp type signal while spreading noise. In other words, a JTFA may increase the SNR for a chirp type signal. There have been some studies on this subject, for example, [6,7]. Most previous studies on the SNR are based on the maximal peaks or the line integrations in the JTFA domain as signals, which may not optimally reflect the signals in the JTFA domain. In a JTFA domain, some areas rather than only points may be signals. Based on this observation, a 3dB SNR definition was recently used in [4,5] and a quantitative SNR analysis for the discrete-time STFT and pseudo Wigner-Ville distribution were obtained. In both cases, rectangular window functions were applied. The SNR obtained in [4,5] is proportional to the sampling rate for multi-component signals.

In this paper, we present a quantitative SNR analysis of linear chirp signals of finite duration in the continuous-time STFT domain with Gaussian window functions. We determine a relationship between the SNRs in the STFT, frequency, and time domains in terms of the chirp rates. The 3dB SNR proposed in [4,5] is used in this paper.

The reason for only considering linear chirps in this paper is because linear chirps often occur in ISAR and SAR applications, see for example [11,12]. In these applications, (i) the transmitted signals may be linear chirps, and (ii) even after the range compression, the radar return signals may still be linear chirps when targets moves with constant accelerations. Because of this, JTFA has been used in ISAR for improving the image resolution of maneuvering targets [1-3].

2 SNR Analysis

We first give the 3dB SNR definition used in [4,5] and then analyze the SNR in the STFT and frequency domains for linear chirp signals.

Consider a signal $s(\Omega)$ corrupted by an additive noise:

$$y(\Omega) = s(\Omega) + n(\Omega), \tag{2.1}$$

where Ω is a variable, such as time t, or frequency f, or joint time-frequency (t, f), and $n(\Omega)$ is an additive noise with 0 mean and variance σ^2. The 3dB SNR is defined as follows.

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Definition 1 *For a signal* $s(\Omega)$, *let*

$$\mathcal{B} \triangleq \{\Omega_1 : \ |s(\Omega_1)|^2 \geq 0.5 \max_{\Omega} |s(\Omega)|^2\}, \quad (2.2)$$

Then, the 3*dB SNR is defined as*

$$SNR \triangleq \frac{\int_{\mathcal{B}} |s(\Omega)|^2 d\Omega}{|\mathcal{B}|\sigma^2}, \quad (2.3)$$

where $|\mathcal{B}|$ *denotes the measure of the set* $\mathcal{B}$.

We first consider a single linear chirp signal model:

$$y(t) = s(t) + n(t) = A\exp(-j(\omega_0 t + \frac{\omega_1}{2}t^2)) + n(t), \quad (2.4)$$

where $\frac{T}{2} \leq t \leq \frac{T}{2}$, A is the signal amplitude, and $n(t)$ is the additive white noise with the following correlation function

$$R_n(t,\tau) = E(n(t)n^*(\tau)) = \sigma^2\delta(t-\tau), \quad (2.5)$$

where $-\frac{T}{2} \leq t, \tau \leq \frac{T}{2}$. Consider the STFT with the Gaussian window function

$$g_\alpha(t) = (\frac{\alpha}{\pi})^{1/4}\exp(-\frac{\alpha}{2}t^2), \ \ \alpha > 0, \quad (2.6)$$

where α is a parameter. Notice that the above Gaussian window function is optimal in terms of the TF localization from the uncertainty principle [9,10]. The STFT of a signal $x(t)$ is

$$STFT_x(t,\omega) = \int_{-\infty}^{\infty} x(\tau)g_\alpha(\tau - t)\exp(-j\tau\omega)d\tau, \quad (2.7)$$

where $x(t)$ can be either $s(t)$ or $n(t)$ in this context. Thus, see for example [9,10],

$$|STFT_x(t,\omega)|^2 = \int\int WVD_x(u,v)\cdot WVD_{g_\alpha}(t-u,\omega-v)dudv, \quad (2.8)$$

where WVD stands for the Wigner-Ville distribution. Since the STFT is a linear transformation, we consider the STFT for the signal $s(t)$ and the noise $n(t)$ separately.

It is not hard to see that the WVD of the above $s(t)$ is

$$WVD_s(t,\omega) = A^2 \ \mathrm{sinc}[\frac{\omega - \omega_1 t - \omega_0}{T}]. \quad (2.9)$$

Since the radar pulse repetition frequency is high enough such that we have high enough sampling rate in the time interval $[-T/2, T/2]$, (2.9) can be approximated as

$$WVD_s(t,\omega) \approx A^2\delta(\omega - \omega_1 t - \omega_0). \quad (2.10)$$

The WVD of the window function is, see for example [11-12],

$$WVD_{g_\alpha}(t,\omega) = 2\exp(-(\alpha t^2 + \frac{1}{\alpha}\omega^2)). \quad (2.11)$$

Thus, the STFT of the signal is

$$|STFT_s(t,\omega)|^2 = 2A^2\int\int \delta(v - \omega_1 u - \omega_0) \cdot \exp(-(\alpha(t-u)^2 + \frac{1}{\alpha}(\omega - v)^2))dudv = \frac{2A^2\sqrt{2\pi}}{\sqrt{2(\alpha + \frac{1}{\alpha}\omega_1^2)}}\exp\left(-\frac{(\omega - t\omega_1 - \omega_0)^2}{\alpha + \frac{1}{\alpha}\omega_1^2}\right). \quad (2.12)$$

In this case, the maximum of $|STFT_s(t,\omega)|^2$ is reached when $\omega = t\omega_1 + \omega_0$ and the maximum is

$$\max_{t,\omega} |STFT_s(t,\omega)|^2 = \frac{2A^2\sqrt{\pi}}{\sqrt{\alpha + \frac{1}{\alpha}\omega_1^2}}. \quad (2.13)$$

Therefore, the 3dB mean of $|STFT_s(t,\omega)|^2$ in the SNR definition (2.2)-(2.3) is

$$\mathrm{mean}_{(t,\omega)\in\mathcal{S}}|STFT_s(t,\omega)|^2, \quad (2.14)$$

where

$$\mathcal{S} = \{(t,\omega) : \ |STFT_s(t,\omega)|^2 > 0.5\frac{2A^2\sqrt{\pi}}{\sqrt{\alpha + \frac{1}{\alpha}\omega_1^2}}\}. \quad (2.15)$$

By some detailed computation and using (2.12)

$$\mathcal{S} = \{|\omega - t\omega_1 - \omega_0|^2 < (\alpha + \frac{1}{\alpha}\omega_1^2)ln2\}.$$

Thus, the 3dB mean signal power is

$$\mathrm{mean}_{(t,\omega)\in\mathcal{S}}|STFT_s(t,\omega)|^2$$

$$= \frac{2A^2\sqrt{\pi}}{\sqrt{\alpha + \frac{1}{\alpha}\omega_1^2}} \frac{1}{\sqrt{ln2}} \int_0^{\sqrt{ln2}} \exp(-u^2) du. \quad (2.16)$$

After the 3dB mean signal power is calculated, let us calculate the mean noise power. Since the noise $n(t)$ is stationary, its mean power can be calculated in the sample space as follows. Using (2.5) we have

$$E|STFT_n(t,\omega)|^2 =$$

$$E\left|\int_{-\infty}^{\infty} n(s) g_\alpha(s-t) \exp(-js\omega) ds\right|^2$$

$$= \sigma^2 \int |g_\alpha(s)|^2 ds = \sigma^2. \quad (2.17)$$

Therefore, by (2.16)-(2.17) the SNR in the STFT domain is

$$SNR_{tf} = \frac{2a\sqrt{\pi}A^2}{\sqrt{\alpha + \frac{1}{\alpha}\omega_1^2}\sigma^2} = \frac{2a\sqrt{\pi}}{\sqrt{\alpha + \frac{1}{\alpha}\omega_1^2}} SNR_t, \quad (2.18)$$

where $SNR_t = A^2/\sigma^2$ is the SNR in the time domain, and

$$a = \frac{1}{\sqrt{ln2}} \int_0^{\sqrt{ln2}} \exp(-u^2) du \approx 0.8. \quad (2.19)$$

The maximum of the SNR_{tf} in terms of the parameter α in the STFT window function $g_\alpha(t)$ in (2.6) is reached when

$$\alpha = |\omega_1|, \quad (2.20)$$

and the maximum is

$$SNR_{tf}^{\max} = \max_\alpha SNR_{tf} = 0.8 \frac{\sqrt{2\pi}}{\sqrt{|\omega_1|}} \cdot SNR_t. \quad (2.21)$$

For multiple linear chirp signal model with K components:

$$\tilde{s}(\tau) = \sum_{i=1}^{K} s_i(\tau) + n(\tau), \quad (2.22)$$

due to the STFT linearity, it is not hard to see that the maximum of the SNR_{tf} in terms of α is bounded such that

$$0.8 \frac{\sqrt{2\pi}}{\sqrt{\max_{1\le i\le K}|\omega_i|}} \cdot SNR_t \le SNR_{tf}^{\max} = \max_\alpha SNR_{tf}$$

$$\le 0.8 \frac{\sqrt{2\pi}}{\sqrt{\min_{1\le i\le K}|\omega_i|}} \cdot SNR_t, \quad (2.23)$$

i.e., bounded between the maximum and the minimum of the components in (2.21). Clearly, when

$$\max_{1\le i\le K} |\omega_i| < 1.28\pi,$$

we have

$$SNR_{tf}^{\max} > SNR_t, \quad (2.24)$$

i.e., the SNR in the STFT domain is greater than the SNR in the time domain. The SNR formulas in (2.24) also imply that, when the absolute values of the coefficients ω_i in $s_i(t)$, i.e., the accelerations of the scatterers, are not too large, the SNR in the STFT domain is greater than the one in the time domain.

It is known that, when the coefficients ω_i in $s_i(t)$ are small, the bandwidth of $s(t)$ may not be large. In other words, the 3dB SNR, SNR_f, in the Fourier transform domain may be also greater than the one in the time domain. This raises the question: which is better between SNR_{tf} and SNR_f? We next want to compare these two SNRs in the STFT domain and in the Fourier transform domain. The 3dB mean power of the Fourier transform $S_i(f)$ of $s_i(t)$ is

$$\frac{\text{Energy of } S_i}{2B_i},$$

where B_i is the bandwidth of the truncated signal s_i, i.e., $B_i = |\omega_i|T$, [11]. Since the Fourier transform preserves the signal energy, the 3dB SNR in the Fourier transform of $s_i(t)$ is

$$SNR_f = \frac{TA^2}{2B_i\sigma^2},$$

where A^2 is the power of s_i. This provides the SNR in the frequency domain for the LFM s_i:

$$SNR_f = \frac{1}{2|\omega_i|} \cdot SNR_t. \quad (2.25)$$

Therefore, we have

$$SNR_{tf}^{\max} = \frac{0.8\sqrt{2\pi}}{\sqrt{|\omega_i|}} \cdot SNR_t = 1.6\sqrt{2\pi}\sqrt{|\omega_i|} \cdot SNR_f. \quad (2.26)$$

This result implies that, when

$$|\omega_i| > 1/(1.6\sqrt{2\pi})^2 = 0.0622, \quad (2.27)$$

we have

$$SNR_{tf}^{\max} > SNR_f. \quad (2.28)$$

3 Simulation

We next want to see a simulation result to show the theoretical result in (2.26). The theoretical SNR ratio $SNR_{tf}^{\max}/SNR_f = 1.6\sqrt{2\pi|\omega_1|}$, solid line, and its simulation, marked by $*$, are shown in Fig. 1, where $T = 200$ is used. The test chirp signal is

$$s(t) = \exp(-j(50t + 3t^2/2)), \ t \in (-100, 100).$$

4 Conclusion

In this paper, we analyzed the SNR in the continuous-time STFT domain with Gaussian windows for linear chirp signals of finite length. It was compared with the SNRs in the time and the frequency domains.

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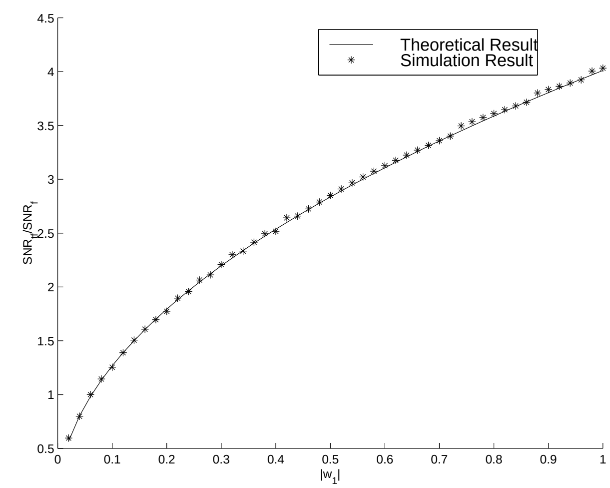

Figure 1: The SNR ratio $SNR_{tf}^{\max}/SNR_f$.