A FRACTIONAL GABOR TRANSFORM

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ABSTRACT

We present a fractional Gabor expansion on a general, non-rectangular time-frequency lattice. The traditional Gabor expansion represents a signal in terms of time and frequency shifted basis functions, called Gabor logons. This constant-bandwidth analysis results in a fixed, rectangular time frequency plane tiling. Many of the practical signals require a more flexible, non-rectangular time-frequency lattice for a compact representation. The proposed fractional Gabor expansion uses a set of basis functions that are related to the fractional Fourier basis and generate a non-rectangular tiling. The completeness and bi-orthogonality conditions of the new Gabor basis are discussed.

1. INTRODUCTION

Time–frequency (TF) analysis provides a characterization of signals in terms of joint time and frequency content [1, 2]. One of the fundamental issues in the TF analysis is obtaining the distribution of signal energy over joint TF plane with a delta function concentration [2]. The Gabor expansion is one of the TF analysis methods which represents a signal in terms of time and frequency translated basis functions called TF atoms [3, 4, 5]. Gabor basis functions $h_{m,k}(t)$ are obtained by shifting and modulating with a sinusoid a single window function $h(t)$, which results in a fixed and rectangular TF plane tiling. Many of the practical signals such as speech, music, biological, and seismic signals however, have time-varying frequency nature that is not appropriate for sinusoidal analysis [6, 7, 8]. Thus the Gabor expansion of such signals will require large number of Gabor coefficients yielding a poor TF localization. The compactness of the Gabor representation is improved if the basis functions match the time-varying frequency behavior of the signal [7, 8, 9, 10]. Here we present a fractional Gabor expansion on a more flexible, non–rectangular TF lattice. The basis functions of the proposed expansion are related to the fractional Fourier basis.

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2. PRELIMINARIES

In this section, we briefly present the traditional Gabor expansion, and the fractional Fourier transform. We give an introduction to the fractional Fourier series expansion.

2.1. The Gabor Expansion

The traditional Gabor expansion [3, 4, 5] represents a signal in terms of time and frequency shifted basis functions, and has been used in various applications to analyze the time–varying frequency content of a signal [7, 11, 12]. Basis functions of the Gabor representation are obtained by translating and modulating with sinusoids a single window function, resulting in a fixed and rectangular TF sampling lattice. An example of such a sampling geometry is shown in Fig. 1. The Gabor expansion of a continuous–time signal $x(t)$ is given by [4, 5]

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{m,k} g_{m,k}(t) \tag{1}$$

where the basis function

$$g_{m,k}(t) = g(t - mT)e^{j\Omega kt} \tag{2}$$

and T is the linear time-shift parameter, and Ω is the frequency sampling interval. The synthesis window function $g(t)$, is normalized to unit energy for definiteness [4]. Existence, uniqueness, convergence and numerical stability of the expansion depend on the choices of parameters T and Ω: critically sampled case is obtained when $\Omega T = 2\pi$. $\Omega T < 2\pi$ is called over-sampling which results in redundancy in the Gabor coefficients, and $\Omega T > 2\pi$ is called under-sampling which causes a loss of information [4].

In general, the set of time and frequency shifted window functions $\{h_{m,k}(t)\}$ forms a non–orthogonal basis for the square–summable continuous functions space $L_2(\mathcal{R})$.

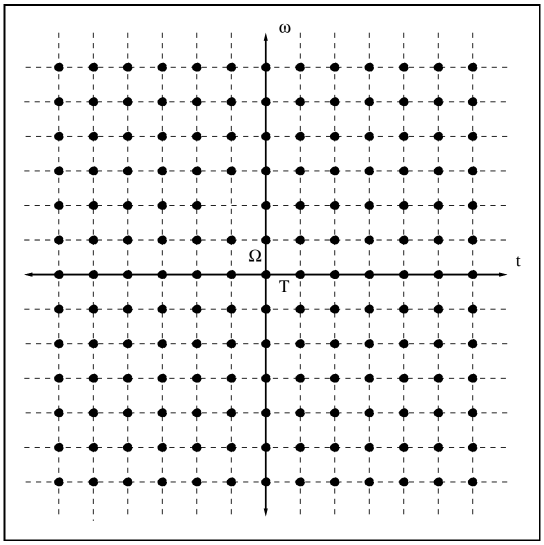

Fig. 1. Rectangular time-frequency plane tiling used in the Gabor expansion.

Hence the calculation of the Gabor coefficients is not a simple task since projection by the usual inner product cannot be used. One of the methods [13], introduced by Bastiaans [14], uses an auxiliary function $\gamma(t)$ called the biorthogonal window or dual function of $g(n)$. Then the Gabor coefficients $\{a_{m,k}\}$ can be evaluated by

$$a_{m,k} = \int_{-\infty}^{\infty} x(t)\ \gamma^*_{m,k}(t)\ dt \qquad (3)$$

where the analysis functions are

$$\gamma_{m,k}(t) = \gamma(t - mT)\ e^{j\Omega kt}. \qquad (4)$$

Completeness condition of the basis set is obtained by substituting (3) into (1) to get that

$$\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g_{m,k}(t)\gamma^*_{m,k}(t') = \delta(t - t') \qquad (5)$$

where $\delta(\cdot)$ denotes the Dirac delta function. The above completeness relation yields equivalent but simpler biorthogonality condition between the analysis and synthesis basis sets via the Poisson-sum formula [4]:

$$\frac{2\pi}{\Omega} \sum_{m=-\infty}^{\infty} g(t - mT)\gamma^* \left(t - \left[m + k\frac{2\pi}{\Omega T} \right] T \right) = \delta_k \quad (6)$$

where $k = 0, \pm1, \pm2, \cdots$, and the factor $\frac{2\pi}{\Omega T}$ is a measure of over-sampling.

In the last decade, the solution of the analysis function for the critical and the over-sampled cases have been given for both continuous- and discrete-time signals [4, 5]. In recent works, Gabor expansion on a non–rectangular TF grid has attracted a considerable attention [7, 8, 15]. A non–rectangular lattice is more appropriate for the TF analysis of signals with time–varying frequency content. Thus the motivation for a fractional Gabor signal expansion.

2.2. Fractional Fourier Transform and Fractional Fourier Series

The Fractional Fourier Transform (FRFT) of a continuous–time signal $x(t)$ is given as [17]

$$\begin{aligned} X_\alpha(u) &= \int_{-\infty}^{\infty} x(t)K_\alpha(t,u)dt = \sqrt{\frac{1 - j\cot\alpha}{2\pi}} \\ &\times \quad e^{\frac{u^2}{2}\cot\alpha} \int_{\infty}^{-\infty} x(t) e^{\frac{t^2}{2}\cot\alpha - jut\csc\alpha} dt \end{aligned}$$

Here $K_\alpha(t,u)$ is the kernel function and it reduces to the classical Fourier kernel for $\alpha = \pi/2$ [17]. In [16] the following fractional Fourier series is given using a set of basis functions similar to the FRFT kernel

$$x(t) = \sum_{k=-\infty}^{\infty} c_{\alpha,k}\ \phi_{\alpha,k}(t) \qquad t \in [-T/2,\ T/2]$$

where $c_{\alpha,k}$ are the fractional Fourier series coefficients. The basis functions, $\phi_{\alpha,k}(t)$, form an orthogonal basis over the range $[-T/2, T/2]$ and they are chosen to be impulses in the α fractional domain:

$$\phi_{\alpha,k}(t) = \sqrt{\frac{\sin\alpha + j\cos\alpha}{T}} e^{-j\frac{1}{2}[t^2 + (k\frac{2\pi}{T}\sin\alpha)^2]\cot\alpha + jk\frac{2\pi}{T}t}$$

where $k = 0, \pm1, \pm2, \ldots,$ controls the frequency sampling. The instantaneous frequencies of these basis functions can easily be obtained as

$$\omega_{\alpha,k}(t) = -t\ \cot\alpha\ +\ k\frac{2\pi}{T} \qquad (7)$$

which are linear functions of time. Hence these basis functions can be used to tile the TF plane in a non–rectangular fashion.

In the next section, we define a Gabor expansion on a non–rectangular TF lattice by means of basis functions with linear instantaneous frequencies.

3. FRACTIONAL GABOR TRANSFORM

We obtain the fractional Gabor expansion by using basis functions with linear instantaneous frequencies, instead of the usual sinusoidal Gabor kernel. The fractional Gabor expansion of a signal $x(t)$ is given by

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{m,k,\alpha} g_{m,k,\alpha}(t) \quad (8)$$

where $g_{m,k,\alpha}(t)$, synthesis basis functions, are given by

$$g_{m,k,\alpha}(t) = g(t - mT)\, W_{\alpha,k}(t) \quad (9)$$

and the fractional kernel is,

$$W_{\alpha,k}(t) = \exp\left\{ j \left[-\frac{1}{2}\left(t^2 + (k\Omega \sin\alpha)^2\right)\cot\alpha + k\Omega t \right] \right\}.$$

where Ω and T are the time and frequency sampling steps, respectively, and $0 \leq \alpha \leq 2\pi$. The basis functions $g_{m,k,\alpha}(t)$ generated by using the above fractional kernel have the linear instantaneous frequency given in (7). Hence the set of fractional basis functions $\{g_{m,k,\alpha}(t)\}$ generates a parallelogram shaped TF sampling lattice shown in Fig. 2. The fractional Gabor coefficients, $a_{m,k,\alpha}$ can be calculated as before by

$$a_{m,k,\alpha} = \int_{-\infty}^{\infty} x(t)\ \gamma^*_{m,k,\alpha}(t) dt \quad (10)$$

where $\gamma_{m,k,\alpha}(t)$ are the analysis functions,

$$\gamma_{m,k,\alpha}(t) = \gamma(t - mT)\, W_{\alpha,k}(t) \quad (11)$$

and they are calculated to be biorthogonal to $g_{m,k,\alpha}(t)$ synthesis functions. When we have $\alpha = \frac{\pi}{2}$, equation (8) reduces to the classical Gabor expansion given in (1). Then the traditional Gabor expansion can be thought as a special case of the fractional expansion. In the following, we explore the completeness and the biorthogonality conditions of the fractional case.

3.1. Completeness of the Fractional Basis

Completeness condition of the fractional Gabor expansion can be obtained by substituting equation (10) into (8) as

$$\sum_{m,k} g_{m,k,\alpha}(t)\gamma^*_{m,k,\alpha}(t') = \delta(t - t') \quad (12)$$

Substituting for the analysis and synthesis functions from (9) and (11), we obtain the explicit condition to be

$$\sum_{m,k} g(t - mT)\gamma^*(t' - mT)\ \times$$
$$\exp\left\{ j \left[\frac{1}{2}\left(t'^2 - t^2\right)\cot\alpha + k\Omega(t - t') \right] \right\} = \delta(t - t')$$

It is clear that for $\alpha = \frac{\pi}{2}$, above condition simplifies to the completeness condition of the traditional Gabor expansion given in (5).

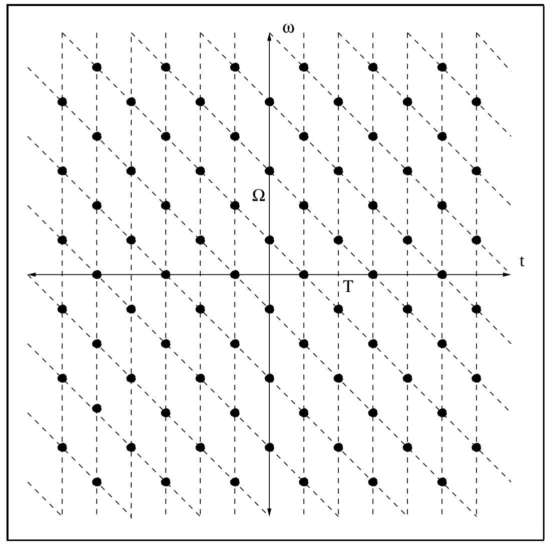

Fig. 2. Time-frequency plane tiling used in the fractional Gabor expansion.

3.2. Fractional Biorthogonality Condition

Now we obtain the biorthogonality condition that the fractional analysis and synthesis function sets must satisfy. The completeness condition in (13) can be rewritten as

$$\sum_m g(t - mT)\gamma^*(t' - mT)\ \exp\left\{ j\frac{1}{2}\left(t'^2 - t^2\right)\cot\alpha \right\}$$
$$\times \sum_k \exp\{jk\Omega(t - t')\} = \delta(t - t')$$

Applying the Poisson-sum formula to the k-summation [4], we obtain

$$\sum_k \exp\{jk\Omega(t - t')\} = \frac{2\pi}{\Omega} \sum_k \delta(t - t' - k\frac{2\pi}{\Omega}) \quad (13)$$

Substituting (13) into (13) yields

$$\frac{2\pi}{\Omega} \sum_m g(t - mT)\gamma^*\left(t - \left[m + k\frac{2\pi}{\Omega T}\right]T\right)$$
$$\times \exp\left\{ j\frac{1}{2}\left[\left(t - k\frac{2\pi}{\Omega}\right)^2 - t^2\right]\cot\alpha \right\}$$
$$\times \sum_k \delta(t - t' - k\frac{2\pi}{\Omega}) = \delta(t - t')$$

We conclude from the above equation that the fractional biorthogonality condition is

$$\frac{2\pi}{\Omega}\exp\left\{j\frac{2k\pi}{\Omega}\left[\frac{k\pi}{\Omega}-t\right]\cot\alpha\right\}\times$$

$$\sum_m g(t-mT)\gamma^*\left(t-\left[m+k\frac{2\pi}{\Omega T}\right]T\right) = \delta_k \quad (14)$$

$m,k = 0,\pm1,\pm2,\cdots$. Notice that the exponential term in the above equation is due to the α-fractional basis, and for $\alpha = \frac{\pi}{2}$, we obtain the biorthogonality condition of the Gabor expansion given in (6). This shows that the fractional Gabor expansion in (8) is the generalization of the usual Gabor expansion into a non–rectangular time–frequency grid.

The analysis window $\gamma(t)$ necessary to find the fractional Gabor coefficients is calculated by solving the linear equation system obtained from (14). Then the analysis set $\{\gamma_{m,k,\alpha}(t)\}$ is used in equation (11) to calculate the fractional Gabor coefficients $a_{m,k,\alpha}$.

4. RESULTS AND DISCUSSION

In this paper, we present a new fractional Gabor expansion for the time–frequency representation of chirp signals. The new representation tiles the TF plane in parallelogram shapes which clearly is a better way of representing chirp signals than the traditional rectangular grid. The basis functions of the fractional expansion are related to the kernel of the fractional Fourier transform and they are impulses in the fractional domain. The biorthogonality relation between the synthesis and analysis functions for the proposed expansion is derived.

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