All-Pass Digital Filter Design in the Frequency-Delay Domain Using the Iterative Quadratic Maximum Likelihood Algorithm

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Abstract **– A new domain, termed the frequency-delay domain, is used to design stable, all-pass digital filters resembling a given delay response in the least-squares sense. This spectral technique identifies the delay response of a stable, second-order, all-pass digital filter as a double sideband suppressed carrier amplitude modulated signal in the frequency-delay domain. Iterative maximum likelihood techniques are used to render the filter coefficients. The algorithm is a significant improvement over related methods because it results in a physically realizable stable all-pass filter that closely approximates a desired delay response.**

1. INTRODUCTION

The design of all-pass digital filters to satisfy a specific phase or group delay function is described in [1]. All-pass filters are frequently used for phase/delay compensation because they do not introduce amplitude distortion as Finite Impulse Response (FIR) or Infinite Impulse Response (IIR) digital filters may do. In classical design methods for FIR and IIR filters, there is no well-established method for all-pass filter design, although eigenfilter algorithms have been successfully employed [2]. The existing methods are not guaranteed to converge nor produce stable filters. This paper describes a method that uses cascaded, second-order, all-pass digital filters [3][4] whose parameters are restricted in such a way, that convergence and stability can be achieved.

2. ALL-PASS DIGITAL FILTERS

All-pass digital filters have unity magnitude response across the entire frequency band. This property introduces a nonlinear relationship between the filter parameters and the phase response, thereby limiting the filter design degrees of freedom.

The Z-transform transfer function of an N^{th} order all-pass digital filter can be expressed as

$$A(z) \quad = \quad \frac{a_N + a_{N-1}z^{-1} + \cdots + a_1 z^{-(N-1)} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_{N-1} z^{-(N-1)} + a_N z^{-N}} . \qquad (1)$$

The transfer function of an all-pass filter of order $2N$ can be described as the product of the transfer function of N second-order all-pass filters that have been cascaded. If we designate the overall gain as M, the composite transfer function can be written as

$$H(e^{j\omega}) = \prod_{i=0}^{N} H_i(e^{j\omega}) = Me^{j\varphi(\omega)} , \qquad (2)$$

where $H_i(e^{j\omega})$ is the transfer function of the i^{th} second-order all-pass unit, and $\varphi(\omega)$ is the composite phase function. Each second-order all-pass unit will have a gain M_i and a phase function $\varphi_i(\omega)$. Each second-order unit can be defined in terms of two polynomial coefficients, a_i and b_i, and the transfer function of unit i is given as

$$H_i(e^{j\omega}) = \frac{a_i + b_i e^{-j\omega} + e^{-j2\omega}}{1 + b_i e^{-j\omega} + a_i e^{-j2\omega}} = M_i e^{j\varphi_i(\omega)} . \qquad (3)$$

With the transfer function of unit i defined in Eq. (3) in terms of a_i and b_i, it is straightforward to show that $\left|H_i(e^{j\omega})\right| = M_i = 1$, for all ω. The composite group delay $\tau_g(\omega)$ is given as

$$\tau_g(\omega) = -\frac{d\varphi(\omega)}{d\omega} = -\sum_{i=1}^{N} \frac{d\varphi_i(\omega)}{d\omega} = \sum_{i=1}^{N} \tau_i(\omega) , \qquad (4)$$

where the group delay of each unit is $\tau_i(\omega) = -d\varphi_i(\omega)/d\omega$. After some algebraic manipulations, the unit group delay can be expressed as

$$\tau_i(\omega, a_i, \zeta_i) = \frac{(1-a_i)}{1 + a_i + 2\sqrt{a_i}\,\cos(\omega - \zeta_i)} + \frac{(1-a_i)}{1 + a_i + 2\sqrt{a_i}\,\cos(\omega + \zeta_i)} , \qquad (5)$$

where ζ_i is termed the frequency-shift parameter and

$$\cos(\zeta_i) = \frac{-b_i}{2\sqrt{a_i}} . \qquad (6)$$

To ensure stability, the poles of a digital filter must be inside the unit circle in the z-plane. To achieve this, both a_i and b_i must be properly bounded. The poles of the filter are found by solving

$$1 + b_i z^{-1} + a_i z^{-2} = 0 , \qquad (7)$$

and are given by the complex conjugate pair

$$z_p^{-1}, (z_p^{-1})^* = \sqrt{a_i}\left[\left(\frac{-b_i}{2\sqrt{a_i}}\right) \pm j\sqrt{1 - \left(\frac{-b_i}{2\sqrt{a_i}}\right)^2}\,\right] = \sqrt{a_i}\, e^{\pm j\zeta_i} . \qquad (8)$$

From Eq. (8), we see that a_i must be a real, positive value such that $0 < a_i < 1$. This restriction on a_i, combined with Eq. (6), implies that b_i must be bounded, that is,

$$-2\sqrt{a_i} \le b_i \le 2\sqrt{a_i} \; . \tag{9}$$

Adhering to these bounds on a_i and b_i will guarantee stable, second-order, all-pass digital filters that can be used to realize desired group delay characteristics.

3. ALL-PASS FILTER DELAY SIGNATURE AND THE FREQUENCY-DELAY DOMAIN

The form of $\tau_i(\omega)$ in Eq. (5) resembles the spectrum of a Double Side Band Suppressed Carrier Amplitude Modulated (DSB-SC AM) signal with carrier frequency ζ_i. We can define a frequency-delay domain in which the delay is a function of the amplitude parameter a_i and frequency ζ_i. The Fourier transform provides a link between the frequency-delay domain and a time-delay domain. The later domain serves our analysis, but has no physical interpretation. The pseudo-modulation present in the delay response of a second-order all-pass digital filter can be described in the time-delay domain using the inverse Fourier transform, that is,

$$\Im^{-1}\{\tau_i(\omega, a_i, \zeta_i)\} = 2 f(t, a_i)\cos(\zeta_i t) \, , \tag{10}$$

where

$$f(t, a_i) = \Im^{-1}\left\{ \frac{(1-a_i)}{1 + a_i - 2\sqrt{a_i}\cos(\omega)} \right\}. \tag{11}$$

With this formulation, $f(t, a_i)$ takes the role of a "baseband" time-delay function.

A frequency-delay domain expression for the composite group delay is,

$$\tau_g(\omega) = \sum_{i=1}^{N} \left[\frac{(1-a_i)}{1 + a_i - 2\sqrt{a_i}\cos(\omega - \zeta_i)} + \frac{(1-a_i)}{1 + a_i - 2\sqrt{a_i}\cos(\omega + \zeta_i)} \right]. \tag{12}$$

In the time-delay domain, using linearity, we have

$$\Im^{-1}\left\{\tau_g(\omega)\right\} = g_g(t) = 2\sum_{i=1}^{N} f(t, a_i)\cos(\zeta_i t) \, , \tag{13}$$

where $f(t, a_i)$ is given by Eq. (11). The same equations also apply in the discrete-time and discrete-frequency domains. However, since the inverse Fourier transform in Eq. (13) cannot be found explicitly, an approximation is used. Dropping the subscript i for notational convenience, the simplest solution results with the following approximation,

$$\frac{(1-a)}{1 + a - 2\sqrt{a}\cos(\omega)} \approx \frac{2\sqrt{c}}{\omega^2 + c} \, . \tag{14}$$

Now, $\hat{f}(t, a)$ is only one term, that is,

$$\hat{f}(t, a) = e^{-\sqrt{c}|t|}, \qquad 0 < a < 1 \, , \tag{15}$$

where c is positive and real. This corresponds to an estimate of the frequency-delay domain baseband spectrum,

$$\Im\{e^{-\sqrt{c}|t|}\} = \frac{2\sqrt{c}}{\omega^2 + c} \, . \tag{16}$$

The temporal and spectral estimate of $f(t,a)$ is denoted the all-pass filter delay signature. With the approximation given in Eq. (14), that the parameter c depends on a as follows

$$c = -2\sqrt[3]{Q} + \frac{10}{\sqrt[3]{Q}} - 10 \, , \tag{17}$$

where Q is defined as

$$Q = -55 - 45\left[\frac{1+a}{2\sqrt{a}}\right] + 15\sqrt{14 + 22\left[\frac{1+a}{2\sqrt{a}}\right] + 9\left[\frac{1+a}{2a}\right]^2} \, . \tag{18}$$

4. MAXIMUM LIKELIHOOD ESTIMATION OF SECOND-ORDER ALL-PASS FUNCTIONS

An initial estimate for the N frequencies-shift values ζ_i, and the N amplitude values a_i, is found from the desired discrete-frequency delay function $\tau_d(k\Omega_0)$, where $\Omega_0 = 2\pi / L$, and L is chosen to satisfy the dimension of $\tau_g(\omega)$. By taking the Inverse Discrete Fourier Transform (IDFT) of this function, the L-point discrete time-delay domain function, $g_d(n)$, or vector $\mathbf{g}_d$ is obtained, that is,

$$g_d(n) = \mathrm{IDFT}\{\, \tau_d(k\Omega_0) \,\}. \tag{19}$$

The desired response, $\tau_d(k\Omega_0)$, is found as the difference between the maximum delay and the system delay response to be compensated, $\tau_{sys}(k\Omega_0)$, in the band of interest, that is,

$$\tau_d(k\Omega_0) = \max\left(\tau_{sys}(k\Omega_0)\right) - \tau_{sys}(k\Omega_0). \tag{20}$$

This design structure guarantees positive values for the desired delay function in the band of interest $[\omega_0, \omega_1]$. A well-known property of all-pass filters [5], is

$$\int_0^{\pi} \tau_g(\omega)\, d\omega = 2N\pi, \quad (21)$$

where N is the number of second-order all-pass digital filters used. Approximating $\tau_d(\omega)$ with $\tau_g(\omega)$, we can establish a lower bound for N, in the band of interest $[\omega_0, \omega_1]$, as

$$N \geq \frac{\omega_1 - \omega_0}{2\pi} \max[\tau_{sys}(\omega)] - \frac{1}{2\pi}\int_{\omega_0}^{\omega_1} \tau_{sys}(\omega)\, d\omega. \quad (22)$$

4.1 Filter Design Formulation

If we generate L samples of $g_g(t)$, denoted $g_g(n)$, the N amplitude values a_i should be chosen such that $g_g(n)$ closely approximates the L values of the desired discrete time-delay domain function $g_d(n)$. The sampled version of $g_g(t)$ in Eq. (13) can be rewritten in matrix form, using the approximation of Eq. (15), as

$$\mathbf{g}_g = \mathbf{A}(\gamma)\mathbf{s} + \mathbf{n}, \quad (23)$$

where $\mathbf{g}_g$ is the vector with L observations in the time-delay domain, $\mathbf{A}(\gamma)$ is defined as a L x N Vandermode matrix [6] when N second order all-pass filter are used in the design,

$$\mathbf{A}(\gamma) = [\mathbf{a}(\gamma_1) \quad \mathbf{a}(\gamma_2) \quad \cdots \quad \mathbf{a}(\gamma_N)]. \quad (24)$$

Here $\mathbf{A}(\gamma)$ is an array of column vectors $\mathbf{a}$, of length L, whose elements are defined at consecutives sample times and that satisfies

$$\mathbf{a}(\gamma_i) = \mathrm{Re}\{[1 \quad e^{\gamma_i} \quad \cdots \quad e^{(L-1)\gamma_i}]^T\}, \quad (25)$$

where

$$\gamma_i = -\sqrt{c_i} + j\zeta_i \quad (26)$$

is a complex variable containing the parameters of interest. Vector $\mathbf{s}$ holds the weighting coefficients used to improve the overall estimation of $\tau_d(k\Omega_0)$ and $\mathbf{n}$ is a vector representing any approximation error. In this formulation, the vector $\mathbf{s}$, is given by,

$$\mathbf{s} = [s_1 \quad s_2 \quad \cdots \quad s_N]^T, \quad (27)$$

where s_i is forced to be a positive integer denoting a cascaded multiple of the i^{th} second-order all-pass filter time-delay response. When $s_i = 1$, there is only one second-order all-pass filter with parameters γ_i. When $s_i = 2$, two units with parameters γ_i are used in cascade. For simplicity, s_i is rounded down to the nearest integer.

4.2 Estimation by Maximum Likelihood

Considering a Gaussian distribution for the error $\mathbf{n}$, the maximum likelihood estimate of the signal parameters γ and vectors of weighting coefficient $\mathbf{s}$, is obtained by minimizing the expression $\|\mathbf{g}_g - \mathbf{A}(\gamma)\mathbf{s}\|^2$. The Iterative Quadratic Maximum Likelihood (IQML) algorithm [6] provides a mean to find γ by minimizing the squared error function ξ

$$\xi = \|\mathbf{g}_g - \mathbf{A}\hat{\mathbf{s}}\|^2 = \|(\mathbf{I} - \mathbf{A}\mathbf{A}^{\Diamond})\mathbf{g}_g\|^2, \quad (28)$$

where the optimum weighting coefficient vector $\hat{\mathbf{s}}$, in the least squares sense, is found using the Moore-Penrose inverse of $\mathbf{A}(\gamma)$, denoted as $\mathbf{A}^{\Diamond}$, such that

$$\hat{\mathbf{s}} = \mathbf{A}^{\Diamond}\mathbf{g}_g = \frac{\mathbf{A}^H \mathbf{g}_g}{\mathbf{A}^H \mathbf{A}}, \quad (29)$$

where $\mathbf{A}^H$ is the Hermitian of $\mathbf{A}$. The spectral factorization of Eq. (23) can be written as [7]

$$\mathbf{R} = \mathbf{g}_g \mathbf{g}_g^H = \mathbf{U}_s \Lambda_s \mathbf{U}_s + \mathbf{U}_n \Lambda_n \mathbf{U}_n, \quad (30)$$

where $\mathbf{R}$ is the autocorrelation function of $\mathbf{g}_g$, Λ_s is a diagonal matrix with the eigenvalues of $\mathbf{A}$, and Λ_n is also a diagonal matrix with the eigenvalues for the error. The columns of $\mathbf{U}_s$ span the range space of $\mathbf{A}$ whereas those of $\mathbf{U}_n$ span its orthogonal complement (or null-space). The projection operator onto the noise subspace is defined as

$$\Pi_A^{\perp} = \mathbf{U}_n \mathbf{U}_n^H = \mathbf{I} - \mathbf{A}\mathbf{A}^{\Diamond}. \quad (31)$$

The metric to be minimized ξ in Eq. (28), can be rewritten in terms of $\Pi_A^{\perp}$ and $\mathbf{R}$ as:

$$\xi = \|(\mathbf{I} - \mathbf{A}\mathbf{A}^{\Diamond})\mathbf{g}_g\|^2 = \|\Pi_A^{\perp}\mathbf{g}_g\|^2 = \mathbf{g}_g^H \Pi_A^{\perp} \mathbf{g}_g = tr(\Pi_A^{\perp}\mathbf{R}) \quad (32)$$

using the trace property, $\mathbf{y}^H\mathbf{x} = tr(\mathbf{x}\mathbf{y}^H)$. The basic idea behind IQML is to re-parameterize the projection matrix $\Pi_A^{\perp}$ using a basis for the null-space of $\mathbf{A}$. This method derives an FIR filter that best suppresses the data, while making the filter's roots the estimated parameters γ_i. This is done by defining a polynomial $b(z)$ with roots at e^{γ_i}, $i = 1,..,N$ as

$$b(z) = z^N + b_1 z^{N-1} + \ldots + b_N = \prod_{i=1}^{N}(z - e^{\gamma_i}), \quad (33)$$

a vector $\mathbf{b}$ with the FIR filter coefficients

$$\mathbf{b} = [b_N \quad \cdots \quad b_1 \quad 1]^T , \tag{34}$$

and a matrix $\mathbf{B}^H$ of rank $L - N$, with shifted versions of $\mathbf{b}$, such that $\mathbf{B}^H$ and $\mathbf{A}$ are orthogonal ($\mathbf{B}^H \mathbf{A} = 0$), that is,

$$\begin{bmatrix} b_N & b_{N-1} & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & b_N & b_{N-1} & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ e^{\gamma_1} & \cdots & e^{\gamma_N} \\ \vdots & \vdots & \vdots \\ e^{(L-1)\gamma_1} & \cdots & e^{(L-1)\gamma_N} \end{bmatrix} = 0 . \tag{35}$$

It can be shown [6], that the mean square error metric ξ can be minimized by solving quadratic

$$\hat{\mathbf{b}} = \min \mathbf{b}^H \mathbf{C}\, \mathbf{b} , \tag{36}$$

where

$$\mathbf{C} = \mathbf{G}_g^H \left(\mathbf{B}^H \mathbf{B}\right)^{-1} \mathbf{G}_g \tag{37}$$

and

$$\mathbf{G}_g = \begin{bmatrix} g_g(N+1) & g_g(N) & \cdots & g_g(1) \\ g_g(N+2) & g_g(N+1) & \cdots & g_g(2) \\ \vdots & \vdots & \cdots & \vdots \\ g_g(L) & g_g(L-1) & \cdots & g_g(L-N) \end{bmatrix} . \tag{38}$$

Successive iterations of the Rayleigh principle will provide the vector $\hat{\mathbf{b}}$ with the maximum likelihood estimated parameters γ_i.

5. OPTIMIZATION BY MEAN SQUARED MINIMIZATION

Once ξ has converged to a minimum, $\hat{\mathbf{b}}$ provides the initial estimates of the N frequencies-shift values, ζ_i, and the N amplitude values, a_i, of the second-order all-pass digital filters. To improve system compensation with $\tau_g(\omega)$, we seek better estimates for ζ_i and a_i via a minimum mean squared iterative gradient approach [8] adding a genetic algorithm that minimizes the chances of converging to a local minimum. Each all-pass filter is optimized individually for a better estimate of ζ_i and then a better estimate of a_i, iterating the process for smaller ξ. Figure 1 shows a desired delay and the composite effect of 8 second-order all-pass filters found by the IQML technique. Figure 2 shows the desired delay, initial and final optimized composite delays, as well as the 8 second-order all-pass filter delays.

6. CONCLUSION

An algorithm to design all-pass digital filters is presented. This method uses cascaded, second-order, all-pass digital filters whose parameters are bounded to produce physically realizable stable filters. Other methods provide a solution in the mean squared error sense, however such solutions do not always provides a stable filter.

8. REFERENCES

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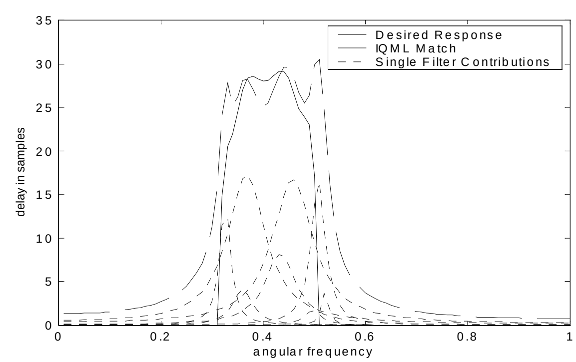

Fig. 1. Desired group delay compared to IQML match and individual filter contributions.

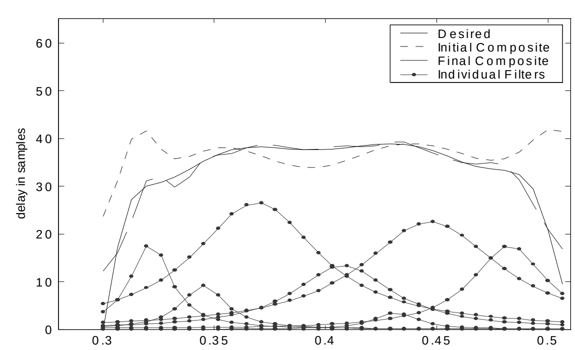

Fig. 2. Group delay comparison after optimization.