DIRECTION FINDING FOR A WAVEFRONT WITH IMPERFECT SPATIAL COHERENCE

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ABSTRACT

We consider the direction-of-arrival (DOA) problem for a wavefront whose amplitude and phase vary randomly along the array aperture. This phenomenon can for instance originate from propagation through an inhomogeneous medium. A simple and accurate DOA estimator is derived in the case of an uniform linear array of sensors. The estimator is based upon a reduced statistic obtained from the sub-diagonals of the covariance matrix of the array output. It only entails computing the Fourier transform of an $(m-1)$-length sequence where m is the number of array sensors. A theoretical expression for the asymptotic variance of the estimator is derived. Numerical simulations validate the theoretical results and show that the estimator has an accuracy very close to the Cramér-Rao bound.

1. INTRODUCTION

Most direction finding methods rely on the assumption that each wavefront impinging on the array is perfectly correlated between sensors and, therefore, contributes a rank-one covariance matrix to the total covariance matrix of the array output vector. However, in many situations this condition is violated. For instance, in underwater acoustics the propagation medium may not be homogeneous giving rise to some coherence loss along the array [1, 2]. As a consequence, the wavefronts undergo random amplitude and phase fluctuations along the array, which from the signal point of view could be considered as a sort of multiplicative noise. A similar phenomenon can be encountered in wireless communications when multiple scatterers in the vicinity of the mobile contribute incoherently to the signal [3, 4]. The methods proposed so far to solve this problem can be classified in three distinct groups. The maximum likelihood estimator (MLE) was derived in [3]; despite its optimal performance, its high computational cost may preclude it from being used in practical situations. Approximate and simpler ML solutions were proposed in [5]. A second group of methods relies on the idea of covariance fitting [2, 3, 6]. This type of estimators enjoy many desirable properties. Since they only use the covariance matrix, they possess some robustness to the lack of knowledge on the data statistics. Additionally, in the Gaussian case, they provide asymptotically efficient estimates. However, they still require the minimization of a multi-dimensional function, which may be computationally intensive and numerically problematic. Finally, many authors advocated the use of subspace-based methods see e.g. [4, 7, 8]. However, since the signal covariance matrix is full-rank even for a single scattered source or a source propagating in an heterogeneous media, conventional subspace-based methods cannot handle properly this problem and hence modifications are required.

In this paper, we introduce a robust and simple estimator without trading-off too much statistical accuracy. Herein, robustness should be understood as the capability to locate the source without much knowledge about the type of coherence loss. Towards this end, we will make as few assumptions on the structure of the multiplicative noise as possible. Furthermore, for both robustness and computational cost issues we consider estimates based on the covariance matrix of the data. In order to simplify the algorithm our estimator will be based on a reduced statistic that bears most of the information regarding the DOA. As will be shown below, this enables us to obtain a simple estimator with an accuracy very close to the Cramér-Rao bound (CRB).

2. DOA ESTIMATION AND PERFORMANCE ANALYSIS

Before presenting the detailed derivation of our estimator, let us formulate the problem and state the hypotheses. We consider a uniform linear array (ULA) of m sensors with inter-element spacing Δ in wavelengths. The received data consists of N independent snapshots $\{\boldsymbol{y}(t)\}_{t=1}^{N}$ which obey the following model

$$\boldsymbol{y}(t) = \boldsymbol{x}(t) \odot \boldsymbol{a}(\theta_0)s(t) + \boldsymbol{n}(t) \tag{1}$$

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where $\boldsymbol{x}(t)$ describes the random multiplicative effect due to the propagation. $s(t)$ is the emitted signal, θ_0 is the DOA and $\boldsymbol{a}(\theta_0)$ denotes the so-called steering vector

$$\boldsymbol{a}(\theta_0) = \begin{bmatrix} 1 & e^{i2\pi\Delta\sin\theta_0} & \cdots & e^{i2\pi(m-1)\Delta\sin\theta_0} \end{bmatrix}^T$$

In (1), $\odot$ stands for the Schur-Hadamard (i.e. element-wise) product and $\boldsymbol{n}(t)$ is assumed to be a zero-mean circularly symmetric Gaussian random vector. The covariance matrix corresponding to (1) can be written as

$$\boldsymbol{R} = \mathcal{E}\left\{\boldsymbol{y}(t)\boldsymbol{y}^H(t)\right\} = \boldsymbol{B}\odot\left[\boldsymbol{a}(\theta_0)\boldsymbol{a}^H(\theta_0)\right] + \sigma_n^2\boldsymbol{I} \quad (2)$$

where $\boldsymbol{I}$ is the identity matrix and σ_n^2 is the noise power in a single sensor. The matrix $\boldsymbol{B}$ models the effects due to the propagation through a random dispersive medium or to the scatterers in the vicinity of the source. For the sake of convenience and without loss of generality, we have absorbed the source power in $\boldsymbol{B}$, i.e. $\boldsymbol{B} = \mathcal{E}\left\{\boldsymbol{x}(t)\boldsymbol{x}^H(t)\right\}\mathcal{E}\left\{|s(t)|^2\right\}$. Note that the signal covariance matrix $\boldsymbol{B}\odot\left[\boldsymbol{a}(\theta_0)\boldsymbol{a}^H(\theta_0)\right]$ in (2) is *full rank* even though only a single source is considered. Our goal is to find a fast and robust method for estimating θ_0 or, equivalently, the spatial frequency $\omega_0 = 2\pi\Delta\sin\theta_0$ since in the field of view $[-90°, 90°]$ these two parameters are related to one another by an invertible mapping.

Before deriving our DOA estimator, a few comments and observations are in order. To gain robustness against non-perfect knowledge of the multiplicative noise characteristics we do not assume, in contrast to most approaches, a specific form for $\boldsymbol{B}$. Herein, we simply assume that $\boldsymbol{B}$ is a real-valued symmetric Toeplitz matrix whose first column is $\widetilde{\boldsymbol{\gamma}} = \begin{bmatrix}\gamma_0 & \gamma_1 & \cdots & \gamma_{m-1}\end{bmatrix}^T$. This assumption is widely accepted for signal propagating through an inhomogeneous medium [1, 2]. In wireless communications, it corresponds to the mild assumption that the scatterers are symmetrically distributed around the mobile [4]. Furthermore, in order to obtain a fast algorithm, we propose to use a reduced-size statistic that concentrates the relevant information about ω_0. More exactly, our approach relies on the following observations. First, note that the (k,ℓ) element of the covariance matrix is

$$\boldsymbol{R}(k,\ell) = \gamma_{|k-\ell|}e^{i(k-\ell)\omega_0} + \sigma_n^2\delta(k,\ell)$$

Hence, each element along the k^{th} sub-diagonal is equal to the same complex number with amplitude γ_k and angle $k\omega_0$ (for $k = 1,\cdots,m-1$). Our intention is to exploit this observation. Note that in the *noiseless case* and assuming that $\boldsymbol{B}(k,\ell) = P\rho_0^{|k-\ell|}$ it was shown in [5] that the maximum likelihood estimator (MLE) of ω_0 is given by

$$\widetilde{\omega}_0 = \text{angle}\left(\sum_{\ell=1}^{m-1}\widehat{\boldsymbol{R}}(\ell+1,\ell)\right)$$

where

$$\widehat{\boldsymbol{R}} = \frac{1}{N}\sum_{t=1}^{N}\boldsymbol{y}(t)\boldsymbol{y}^H(t)$$

denotes the sample covariance matrix. Although it was derived under specific assumptions, $\widetilde{\omega}_0$ was shown to be quite accurate even in the presence of noise, provided that the signal to noise ratio is relatively high. However, its performance degrades at low SNR's or whenever $\boldsymbol{B}(k,\ell) \neq \rho_0^{|k-\ell|}$. An important observation is that $\widetilde{\omega}_0$ uses only the sum of the elements of $\widehat{\boldsymbol{R}}$ along its main sub-diagonal. Here, *we retain this idea of using the sub-diagonals of the covariance matrix* but in a different setup. Indeed, we consider the framework of covariance-based methods and use non-linear least squares fitting, as described next. Let, for $k = 1,\cdots,m-1$

$$\begin{aligned} z_k &= \sum_{\ell=1}^{m-k}\boldsymbol{R}(k+\ell,\ell) \\ &= (m-k)\gamma_k e^{ik\omega_0} \triangleq \zeta_k e^{ik\omega_0} \end{aligned} \quad (3)$$

and let $\widehat{z}_k = \sum_{\ell=1}^{m-k}\widehat{\boldsymbol{R}}(k+\ell,\ell)$ be a consistent estimate of z_k. We propose to estimate ω_0 and $\boldsymbol{\zeta} = \begin{bmatrix}\zeta_1 & \cdots & \zeta_{m-1}\end{bmatrix}^T$ as

$$\begin{aligned} \widehat{\omega}_0, \widehat{\boldsymbol{\zeta}} &= \arg\min_{\omega,\boldsymbol{\zeta}}\sum_{k=1}^{m-1}\left|\widehat{z}_k - \zeta_k e^{ik\omega}\right|^2 \\ &= \arg\min_{\omega,\boldsymbol{\zeta}}\sum_{k=1}^{m-1}\left|\widehat{z}_k e^{-ik\omega} - \zeta_k\right|^2 \end{aligned} \quad (4)$$

Since for any ω the criterion is quadratic with respect to $\boldsymbol{\zeta}$, the solution for $\boldsymbol{\zeta}$ is $\widehat{\zeta}_k = \text{Re}\left[\widehat{z}_k e^{-ik\omega}\right]$. Inserting this value into (4) and after some straightforward manipulations (see [9] for details), the estimate of ω_0 is obtained as

$$\widehat{\omega}_0 = \arg\max_{\omega}\text{Re}\left[\sum_{k=1}^{m-1}\widehat{z}_k^2 e^{-i2k\omega}\right] \quad (5)$$

Therefore, we end up with a very simple expression for the estimate of ω_0 which entails summing along the sub-diagonals of $\widehat{\boldsymbol{R}}$, computing the Fourier transform in (5) and looking for the location of its maximum. Additionally, we reiterate the fact that this method is robust to mismodelling the covariance matrix of the multiplicative noise. Once $\widehat{\omega}_0$ is available, θ_0 is simply estimated as

$$\widehat{\theta}_0 = \arcsin\left(\frac{\widehat{\omega}_0}{2\pi\Delta}\right) \quad (6)$$

We next analyze the statistical performance of the estimator in (5)-(6).

Proposition 1. *Under the assumptions that $\{\boldsymbol{y}(t)\}_{t=1}^{N}$ are independent Gaussian random vectors with a covariance matrix given by (2), the asymptotic variance of $\widehat{\omega}_0$ in (5) is given by*

$$\lim_{N\to\infty} N \mathrm{var}\left(\widehat{\omega}_0\right) \triangleq \lim_{N\to\infty} N \mathcal{E}\left\{\left(\widehat{\omega}_0-\omega_0\right)^2\right\} = \frac{\boldsymbol{\zeta}^T \boldsymbol{D}(\boldsymbol{\Gamma}-\widetilde{\boldsymbol{\Gamma}})\boldsymbol{D}\boldsymbol{\zeta}}{2(\boldsymbol{\zeta}^T\boldsymbol{D}^2\boldsymbol{\zeta})^2} \quad (7)$$

where $\boldsymbol{D}=\mathrm{diag}\left(1,2,\cdots,m-1\right)$ *and*

$$\begin{aligned}\boldsymbol{\Gamma}(k,\ell) = &\sum_{p=1}^{m-k}\sum_{q=1}^{m-\ell}\gamma_{|p-q+k-\ell|}\gamma_{|p-q|} \\ &+ 2\sigma_n^2\gamma_{|k-\ell|}\min(m-\ell,m-k) \\ &+ (m-k)\sigma_n^4\delta(k,\ell)\end{aligned} \quad (8)$$

$$\begin{aligned}\widetilde{\boldsymbol{\Gamma}}(k,\ell) = &\sum_{p=1}^{m-k}\sum_{q=1}^{m-\ell}\gamma_{|p-q+k|}\gamma_{|p-q-\ell|} \\ &+ 2\sigma_n^2\gamma_{k+\ell}\max(m-k-\ell,0)\end{aligned} \quad (9)$$

Proof. see [9] □

Corollary 1. Since there exists a one-to-one continuous and differentiable mapping from ω_0 to θ_0, it follows that

$$\lim_{N\to\infty} N \mathrm{var}\left(\widehat{\theta}_0\right) = \left(\frac{\partial\theta_0}{\partial\omega_0}\right)^2 \lim_{N\to\infty} N\mathrm{var}\left(\widehat{\omega}_0\right) = \frac{\boldsymbol{\zeta}^T \boldsymbol{D}(\boldsymbol{\Gamma}-\widetilde{\boldsymbol{\Gamma}})\boldsymbol{D}\boldsymbol{\zeta}}{2(2\pi\Delta\cos\theta_0)^2(\boldsymbol{\zeta}^T\boldsymbol{D}^2\boldsymbol{\zeta})^2} \quad (10)$$

The previous results provide closed-form expressions for the asymptotic variances of $\widehat{\omega}_0$ and $\widehat{\theta}_0$. These expressions are checked against empirical variances in the next section, and also numerically compared with the CRB.

3. NUMERICAL EXAMPLES

In this section, we illustrate the performance of our estimator by means of Monte-Carlo simulations and validate the theoretical analysis. We consider a ULA with $m = 8$ sensors spaced a half wavelength apart. The DOA of the source is $\theta_0 = 10°$. The signal is given by (1) where $\boldsymbol{x}(t)$ is modeled as a zero-mean Gaussian random vector with covariance matrix $\boldsymbol{B}(k,\ell) = \rho^{|k-\ell|}$ and $\rho = 0.9$ unless otherwise stated. For each figure, 300 Monte-Carlo simulations were run to estimate the root mean-square error (RMSE) of the estimates. For comparison purposes, the Cramér-Rao bound [3] is also displayed. All values are given in degrees (°).

First, the influence of the number of snapshots N is investigated in Figure 1 for $SNR = 0$ dB. It can be observed that the empirical and theoretical results are in very good agreement, even for a small number of snapshots. The RMSE of the estimator is seen to be very close to the CRB.

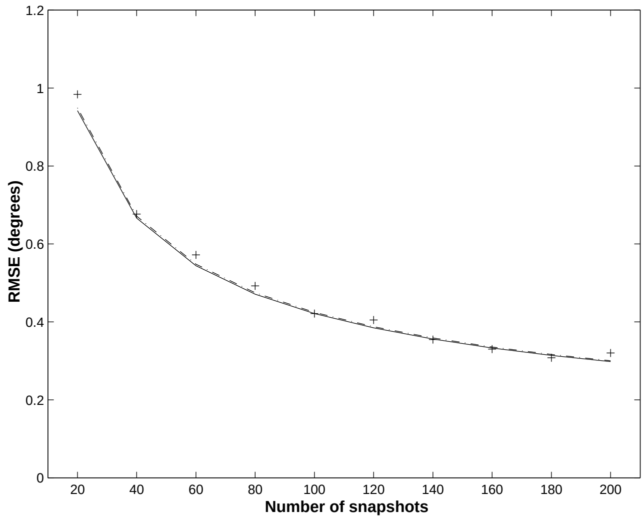

Fig. 1. CRB (solid line), empirical (+) and theoretical (dash-dotted line) RMSE of the DOA estimate versus N. $m = 8$, SNR=0dB and $\rho = 0.9$.

A second series of simulations deals with the influence of the SNR, see Figure 2. The main point to be noted is that the performance remains quite close to the CRB even for low SNR's. When SNR increases, both the theoretical variance of the estimator and the CRB tend to a constant term. In fact, both of them are theoretically of the form $N^{-1}(a+bSNR^{-1}+cSNR^{-2})$ where a, b, c are constants. The term a is zero only if there is no multiplicative noise. Otherwise, the variance does not go to zero as SNR increases since, even when $\sigma_n^2 = 0$, i.e. when there is no additive noise, there is still multiplicative noise which prevents us from retrieving ω_0 exactly.

Next, the influence of the number of sensors is examined in Figure 3. The performance is shown to be very close to the CRB whatever the value of m.

Finally, it is interesting to examine how the estimator behaves as the coherence loss increases, that is when the correlation among the elements is less and less pronounced. In Figure 4 the coherence loss at a wavelength separation, $(20\log_{10}\rho)$, is varied from -3 dB to -0.25 dB. The estimator does not exhibit a breakdown; its performance is only slightly degraded, which is an interesting feature of the method and proves that it can work under severe conditions.

4. CONCLUSIONS

In this paper we have proposed a very simple and robust method for estimating the direction of arrival of a source in the presence of amplitude and phase fluctuations of the

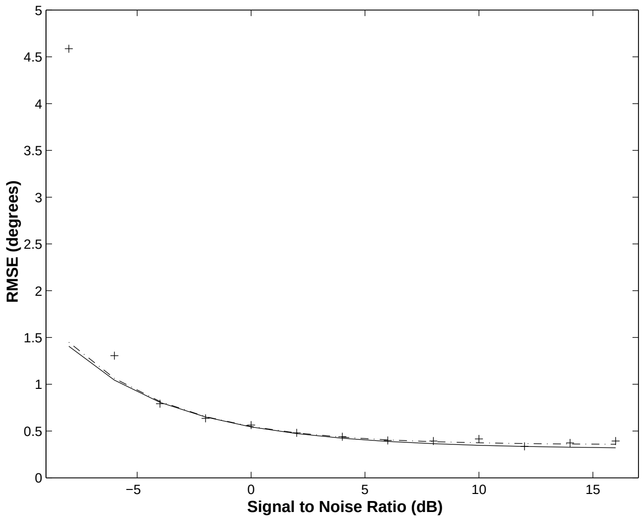

Fig. 2. CRB (solid line), empirical (+) and theoretical (dash-dotted line) RMSE of the DOA estimate versus SNR. $m = 8$, $N = 60$ and $\rho = 0.9$.

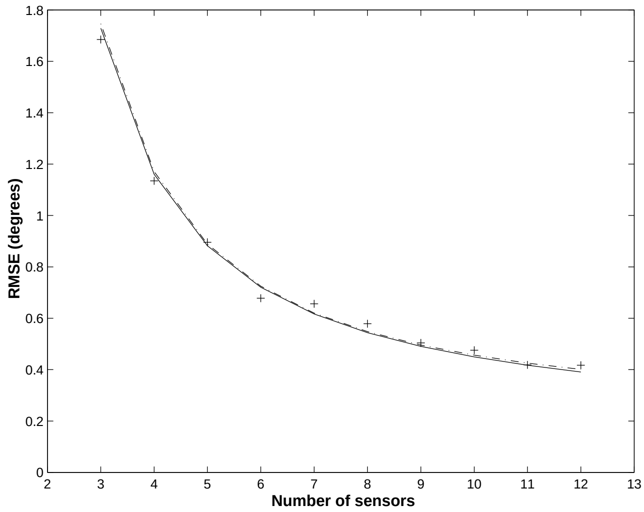

Fig. 3. CRB (solid line), empirical (+) and theoretical (dash-dotted line) RMSE of the DOA estimate versus m. $N = 60$, SNR=0dB and $\rho = 0.9$.

wavefront. It relies on the sum of the sub-diagonal elements of the array output covariance matrix and only entails finding the maximum of a function which can be expressed as a Fourier transform. A theoretical expression for the asymptotic variance of the estimator was derived and its validity was shown through numerical simulations. Numerical examples showed that the proposed estimate has an accuracy very close to the CRB.

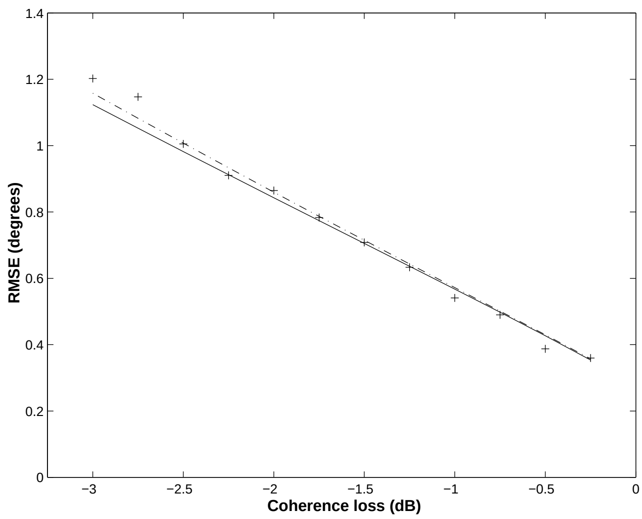

Fig. 4. CRB (solid line), empirical (+) and theoretical (dash-dotted line) RMSE of the DOA estimate versus ρ. $m = 8$, $N = 60$ and SNR=0dB.

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