EXPRESSION OF THE CAPACITY FOR THE GILBERT CHANNEL IN PRESENCE OF INTERLEAVING

T. Rancurel, D. Roviras, and F. Castanié

ENSEEIHT/TESA, 2 rue Camichel, BP 7122, 31071 Toulouse cedex 7, FRANCE
rancurel@len7.enseeiht.fr

ABSTRACT

Many models based on Hidden Markov Models were developed to model errors bursts in communication channels. These models allow the computation of error distributions as well as the capacity expression of a channel. In this paper, we evaluate the effect of interleaving on the capacity of the widely used Gilbert model. An expression of this capacity is derived and is illustrated by simulation. This allows, when having an identified error model, simulation in an easy manner of the effects of any interleaving depth on the capacity of the channel.

1. INTRODUCTION

Due to the increasing complexity of networks protocols, it is important to examine the expected performance with an accurate model of the errors distribution. With this aim, different models based on Hidden Markov Models (HMM) have been developed to model communication systems. These models were successfully used to model errors sources in communication channels, and especially in the context of wireless communication channels where errors occur in bursts. The first model, used to study errors sources in binary communication channels using HMM, is the Gilbert model [1]. Afterwards, models based on more number of states to get a better modelling have been proposed [3] [4] [5] [6] [7].

In this paper, we give an expression for the capacity of the Gilbert channel in the presence of interleaving. This result can be used in the context of bursty channels in order to predict the performance.

We use a Gilbert model as it is quite simple and widely used. Despites its simplicity, this model allows, for example, to model block errors [8] and burst errors in the context of ATM communications[9].

This model is composed of two states: the "bad" state B with a crossover probability p_g and the good state G with a null crossover probability.

Our paper is organized as follows. In section 2, we review the Gilbert model. In section 3, we refer to the expression of the transition probability matrix in the presence of interleaving. The expression of the capacity is derived in section 4. Section 5 gives an expression for the capacity of this model in the presence of interleaving. Simulations are carried out in section 6.

2. THE GILBERT MODEL

The Gilbert model [1] [2] shown in Fig. 1 is a varying binary symmetric channel. For this model, the crossover probabilities are determined by the current state of a discrete-time stationary binary Markov process. It is composed of two states, respectively G for good with a null error probability, and B for bad with a crossover probability p_g. Due to the underlying Markov nature of the channel it has memory that depends on the transition probabilities between the states.

Its state transition probability matrix is:

$$P = \begin{pmatrix} g & 1-g \\ 1-b & b \end{pmatrix} \tag{1}$$

Thus the observation transitional probability matrices are given by:

$$\begin{aligned} P(0) = PF(0) &= \begin{pmatrix} g & (1-g)(1-p_g) \\ 1-b & b(1-p_g) \end{pmatrix} \\ P(1) = PF(1) &= \begin{pmatrix} 0 & (1-g)p_g \\ 0 & bp_g \end{pmatrix} \end{aligned} \tag{2}$$

Let $\Pi = (\pi_G \; \pi_B)$ the steady state probabilities vector. We have:

$$\begin{cases} \pi_G = \frac{1-b}{2-b-g} \\ \pi_B = \frac{1-g}{2-b-g} \end{cases} \tag{3}$$

Then, the average symbol error rate is:

$$p = p_g \pi_B = \frac{p_g(1-g)}{2-b-g}$$

The transition matrix can be rewritten in the following form [10] [11]:

$$P = \begin{pmatrix} \frac{-(p-p_g)}{p_g} + \frac{p}{p_g}\rho_0 & \frac{p}{p_g} - \frac{p}{p_g}\rho_0 \\ \frac{-(p-p_g)}{p_g} + \frac{(p-p_g)}{p_g}\rho_0 & \frac{p}{p_g} - \frac{(p-p_g)}{p_g}\rho_0 \end{pmatrix} \tag{4}$$

with $\rho_0 = \frac{p-bp_g}{p-p_g}$.

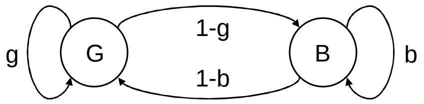

Fig. 1. 2-states Gilbert model

3. EFFECT OF INTERLEAVING

Interleaving is frequently used on bursty channel as mobile ones. It allows randomization of the errors on a codeword. Indeed, the channel coding is most efficient when the bit errors are uniformly distributed within the transmission bit stream.

Consider a sequence of bits divided into l blocks of bits. The transmitter assembles the successive block and transmits the first bit of each block, followed by the second bit of each block, and so on. The receiver performs the opposite de-interleaving process to reproduce the original bit blocks. This is known as an interleaving depth of l.

So, when a code is interleaved to depth $l > 1$, two consecutive symbols of a code word are spaced apart by l symbol times. The corresponding transition matrix P^l can be expressed as [10] [11]:

$$P^l = \begin{pmatrix} \frac{-(p-p_g)}{p_g} + \frac{p}{p_g}\rho_0^l & \frac{p}{p_g} - \frac{p}{p_g}\rho_0^l \\ \frac{-(p-p_g)}{p_g} + \frac{(p-p_g)}{p_g}\rho_0^l & \frac{p}{p_g} - \frac{(p-p_g)}{p_g}\rho_0^l \end{pmatrix}$$

Therefore, for the interleaved channel, we have:

$$g_l = \frac{-(p-p_g)}{p_g} + \frac{p}{p_g}\rho_0^l$$
$$b_l = \frac{p}{p_g} - \frac{(p-p_g)}{p_g}\rho_0^l$$

4. CAPACITY OF THE GILBERT MODEL

The method used in this section is very similar to the one given by Gilbert except that we simplify some computations.

Let the channel output $Y = \{y_i : i = 1, 2, \cdots\}$ be the mod 2 sum of the input $X = \{x_i : i = 1, 2, \cdots\}$ and the noise $Z = \{z_i : i = 1, 2, \cdots\}$, where all three are elements from Galois field $GF(2)$, and it is assumed that Z is independent of X.

The capacity of a binary burst noise channel is given by:

$$C = 1 - H = 1 - H(Y/X) \qquad (5)$$

With $H(Y/X)$ the entropy of the sequence of noise digits:

$$H(Y/X) = \lim_{N\to\infty} \sum_{z_i=0,1} P(z_1, \cdots, z_N) h(z_1, \cdots, z_N) \qquad (6)$$

The definition of $h(z_1, \cdots, z_N)$ corresponds to $H(Z_{N+1}/Z_1 = z_1, \cdots, Z_N = z_N)$:

$$h(z_1, \cdots, z_N) = -\sum_{z_{N+1}=0}^{1} P(z_{N+1}/z_1, \cdots, z_N) \log_2 P(z_{N+1}/z_1, \cdots, z_N)$$

The entropy (6) can be calculated directly from the definition of $h(z_1 \cdots z_n)$. This task can be simplified as there is only one error-state and therefore, in case of error, we can know the original state.

$z_i = 1$ implies that the state at instant i is B, so, we have:

$$h(z_1, \cdots, z_{i-1}, 1, z_{i+1}, \cdots, z_N) = h(1, z_{i+1}, \cdots, z_N) \qquad (7)$$

Therefore, $h(z_1, \cdots, z_N)$ is totally defined by each of the 2^N h's terms in the sum (6) which are one of the $N+1$ terms:

$$h(1),\ h(10),\ h(100), \cdots,\ h(10^k),\ \cdots,\ h(10^{N-1}),\ h(0^N)$$

Let now define the error free run distribution $u_k = \Pr(0^k/1)$, with $u_0 = 1$.

We have:

$$\Pr(0/10^k) = \frac{\Pr(10^{k+1})}{\Pr(10^k)} = \frac{\Pr(0^{k+1}/1)}{\Pr(0^k/1)} = \frac{u_{k+1}}{u_k}$$

which gives using (7):

$$\begin{aligned} h(10^k) &= -\Pr(0/10^k)\log_2 \Pr(0/10^k) \\ &\quad -\Pr(1/10^k)\log_2 \Pr(1/10^k) \\ &= -\frac{u_{k+1}}{u_k}\log_2 \frac{u_{k+1}}{u_k} - (1 - \frac{u_{k+1}}{u_k})\log_2 (1 - \frac{u_{k+1}}{u_k}) \end{aligned}$$

We can then deduce the capacity C:

$$\begin{aligned} C &= 1 - H(Y/X) = 1 - \sum_{k=0}^{\infty} \Pr(10^k) h(10^k) \qquad (8) \\ &= 1 + \sum_{k=0}^{\infty} p(u_k - u_{k+1}) \log_2 (u_k - u_{k+1}) \end{aligned}$$

The definition of the error free run is given by the following formula [12]:

$$u_k = \Pr(0^m|1) = \frac{\Pi P(1) P^m(0) \mathbf{1}}{\Pi P(1) \mathbf{1}}$$

$P^m(0)$ can be calculated by using the following spectral representation:

$$P(0) = T^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T$$

the eigenvalues of $P(0)$ are then:

$$\begin{aligned} \lambda_{1,2} &= \left(\frac{b(1-p_g)+g}{2}\right) \\ &\quad \pm \sqrt{\left(\frac{b(1-p_g)+g}{2}\right)^2 + (1-b-g)(1-p_g)} \end{aligned}$$

The transform matrix T is composed from the matrix $P(0)$ left eigenvectors:

$$T = \begin{pmatrix} 1-b & \lambda_1 - g \\ 1-b & \lambda_2 - g \end{pmatrix}$$

which gives:

$$T^{-1} = \frac{1}{(1-b)(\lambda_2 - \lambda_1)} \begin{pmatrix} \lambda_2 - g & -(\lambda_1 - g) \\ -(1-b) & 1-b \end{pmatrix}$$

So, $P^m(0)$ is:

$$P^m(0) = T^{-1} \begin{pmatrix} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{pmatrix} T$$

And the error free run is finally given by:

$$\begin{aligned} u_k &= \frac{\Pi P(1) P^k(0) \mathbf{1}}{\Pi P(1) \mathbf{1}} \\ &= \frac{1}{\lambda_1 - \lambda_2}((1-g-b+\lambda_1)\lambda_1^k - (1-g-b+\lambda_2)\lambda_2^k) \end{aligned}$$

We then have the following expression for the capacity:

$$\begin{aligned} C &= 1 + \frac{1}{\lambda_1 - \lambda_2} p \sum_{k=0}^{\infty} ((1-g-b+\lambda_1)(1-\lambda_1)\lambda_1^k \\ &\quad -(1-g-b+\lambda_2)(1-\lambda_2)\lambda_2^k) \\ &\quad \log_2 \Big((1-g-b+\lambda_1)(1-\lambda_1)\lambda_1^k \\ &\quad -(1-g-b+\lambda_2)(1-\lambda_2)\lambda_2^k\Big) \end{aligned} \tag{9}$$

5. CAPACITY OF THE INTERLEAVED GILBERT CHANNEL

Using (9) and the results from the previous section, we can deduce the capacity with interleaving:

$$\begin{aligned} C &= 1 + \frac{1}{\lambda_1 - \lambda_2} p \sum_{k=0}^{\infty} ((1-g_l-b_l+\lambda_1)(1-\lambda_1)\lambda_1^k \\ &\quad -(1-g_l-b_l+\lambda_2)(1-\lambda_2)\lambda_2^k) \\ &\quad \log_2 \Big((1-g_l-b_l+\lambda_1)(1-\lambda_1)\lambda_1^k \\ &\quad -(1-g_l-b_l+\lambda_2)(1-\lambda_2)\lambda_2^k\Big) \end{aligned}$$

with:

$$g_l = \frac{-(p-p_g)}{p_g} + \frac{p}{p_g}\rho_0^l$$
$$b_l = \frac{p}{p_g} - \frac{(p-p_g)}{p_g}\rho_0^l$$
$$p = \frac{p_g(1-g)}{2-b-g}$$
$$\rho_0 = \frac{p-bp_g}{p-p_g}$$
$$\lambda_1 = \left(\frac{b_l(1-p_g)+g_l}{2}\right) + \sqrt{\left(\frac{b_l(1-p_g)+g_l}{2}\right)^2 + (1-b_l-g_l)(1-p_g)}$$
$$\lambda_2 = \left(\frac{b_l(1-p_g)+g_l}{2}\right) - \sqrt{\left(\frac{b_l(1-p_g)+g_l}{2}\right)^2 + (1-b_l-g_l)(1-p_g)}$$

p remains the same, as interleaving does not affect the average BER.

6. SIMULATIONS

The aim of this section is to illustrate the previous results. This is done by simulating different interleaving depths.

Consider C^{BSC} and C^{CSI} the capacity of respectively the equivalent memoryless channel and with channel state information. We have:

$$C^{BSC} = 1 - p\log_2 p - (1-p)\log_2(1-p)$$

and

$$\begin{aligned} C^{CSI} &= \pi_B(1 - h(\Pr(z_i = 1/S_i = B))) \\ &\quad + \pi_G(1 - h(\Pr(z_i = 1/S_i = G))) \\ &= \pi_B(1 - h(p_g)) + \pi_G(1 - h(0)) \end{aligned}$$

with:

$$h(p) = -p\log_2 p - (1-p)\log_2(1-p)$$

It can be shown that the following inequality holds:

$$C^{BSC} \leq C \leq C^{CSI}$$

Mushkin [13] proposed a measure of memory. We will use this to illustrate these inequalities. This memory is given by:

$$\mu = g + b - 1$$

Let us also define the good to bad ratio ρ of the channel by:

$$\rho \triangleq \frac{1-b}{1-g}$$

It can be easily shown that C^{BSC} and C^{CSI} are independent of μ as well as of l.

The real capacity of the channel does not depend on the interleaver since this is a reversible operation [13]. However, the computed capacity is reduced with the depth of the interleaving because the channel memory is converted into a latent fragmented form which is not properly used in a conventional decoder, although can be properly used as explained in [13].

For simulations, we fix ρ, and by varying μ in the interval $[0;1]$ we illustrate the results from the previous section.

ρ and p_g are respectively fixed to 3 and 0.5.

The following figure gives a plot of the capacity vs. μ for different interleaving depths:

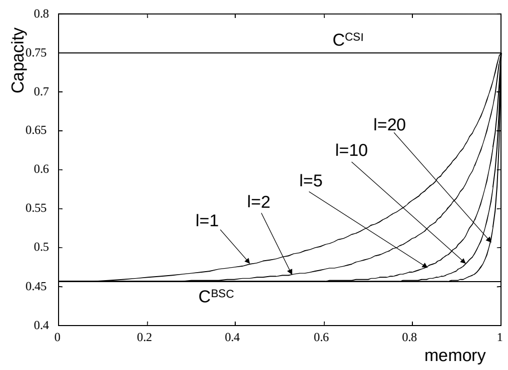

Fig. 2. Capacity variation vs. ρ for different interleaving depths

As expected, the capacity is bounded by C^{CSI} and C^{BSC}.

We can also notice that when the depth of interleaving is long enough, the capacity approaches the capacity of the equivalent memoryless channel.

7. CONCLUSION

In this paper, we studied the effects of interleaving on the capacity of the Gilbert channel. We gave an expression of the capacity as a function of the Gilbert model parameters and the depth of interleaving.

The interleaving depth has not effect on the real capacity of the channel. However, when the interleaving depth increases, the computed capacity decreases and therefore, the effectiveness of a conventional decoder increases.

When studying the effects of interleaving, we can then derive the expression of the capacity in a theoretical way for any interleaving depth as soon as the initial model is defined.

8. REFERENCES

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