FOURTH ORDER NON-DATA-AIDED SYNCHRONIZATION

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ABSTRACT

The study of parameter estimation under additive and multiplicative noise terms constitutes an interesting signal processing research topic with important consequences in the development of high-performance digital communication receivers. In this sense, the paper addresses the analysis of non-assisted digital synchronizers based on up to fourth-order moments and it shows how this alternative outperforms the conventional second-order techniques. The study is performed for *MSK* (Minimum Shift Keying) as a particular case of the binary *CPM* modulations. In the paper this transmission scheme is adopted because it allows a simple extension to any linear digital modulation and to any multiple access modulation, as well. Simulation results show that the higher-order techniques exhibit a parameter estimation variance closer to the so-called Modified Cramer-Rao bound for moderate to high *SNR* if compared to second-order techniques.

1. INTRODUCTION

The study of digital synchronization methods has been an important research area during the last decade ([1] and references herein). Non-Data-Aided (*NDA*) synchronization, i.e. parameter estimation when the transmitted symbols are unknown, is particularly interesting. These techniques have to cope with a multiplicative random term associated with the unknown transmitted symbols as well as the usual thermal *AWGN*.

Most of the *NDA* methods have been proposed based on heuristic reasoning and ad hoc designed for each specific modulation format. Recently, the authors ([2],[3],[4] and references therein) have studied the formulation of any *NDA* synchronization technique for timing and frequency error estimation based on second order moments under a common analysis framework, independently of considering single or multiple access modulations [5].

This work has been supported by: TIC98-0703, TIC99-0849 TIC2000-1025, FIT-070000-2000-649 (CICYT) and 2000SGR-00083 (CIRIT/Generalitat de Catalunya).

The main contributions of these previous works can be summarized in two points. On the one hand, it has been shown that any second-order based *NDA* technique is lower bounded by the Gaussian Unconditional Cramer-Rao Bound (*UCRBG*) despite the non-gaussian nature of the transmitted symbols. On the other hand, the *UCRBG* can be attained by adopting an adequate compression of the Likelihood function.

In [2] and [3] the authors showed that for low *SNRs*, the *UCRBG* and the exact *UCRB* are asymptotically equivalent. In other words, it has been shown that the so-called (optimistic) Modified Cramer-Rao Bound (*MCRB*) [6] may not be attained by a *NDA* method for low *SNRs*. Mengali et al. presented in [1] (Section 9.4) an ad hoc fourth order synchronizer based on intuitive reasoning capable of outperforming the *UCRBG* for moderate to high *SNRs*. This interesting result motivated the present work. The main goal of this paper is the development of a new analysis framework for the design of fourth order *NDA* timing and frequency synchronizers trying to unify the *NDA* synchronization theory under a common perspective. Simulations have shown that the use of high-order statistics significantly improves the digital synchronizer performance for moderate to high *SNRs*.

The structure of the paper is the following. The signal model and the problem statement are presented in Section 2. Section 3 describes the compression of the Likelihood function in order to remove the random term given by the transmitted symbols, also referred to as nuisance parameters. Section 4 contains the most interesting contribution of the paper by describing the (so-called) pseudo-symbols extraction and decorrelation procedure. Simulations results can be found in Section 5 and, finally, conclusions are drawn in Section 6.

2. DISCRETE-TIME SIGNAL MODEL

The complex envelope for the received MSK signal can be expressed as a linear modulation using the Laurent expansion [2] [1]:

$$r(t) = \sum_{n=-\infty}^{\infty} c_n g(t - nT - \tau) + w(t) \qquad (1)$$

and its sampled form is:

$$r(kT_s) = \sum_{n=-\infty}^{\infty} c_n g(kT_s - nT - \tau) + w(kT_s) \quad (2)$$

where $\{c_n\}$ are the *MSK* transmitted pseudo-symbols, $g(t)$ the *MSK* pulse shape, T is the symbol period, T_s is the sampling period, τ is the symbol timing error of the received signal and $w(t)$ is the complex additive white Gaussian noise (AWGN) term with zero mean and variance σ_w^2. Perfect phase and frequency knowledge is assumed. $\{c_n\}$ and $g(t)$ have the following expressions [1]:

$$\begin{aligned} c_n &= e^{j\phi} e^{j\frac{\pi}{2}\sum_{i=0}^{n} a_i} \\ g(t) &= sin(\tfrac{\pi t}{2T}) \end{aligned} \quad (3)$$

where the initial phase ϕ can take the values $\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ and $\{a_i\}$ are the binary *MSK* symbols. In order to simplify the analysis, the pulse $g(t)$ is assumed to pass through the anti-aliasing filter of bandwidth $1/T$ without distortion.

Considering a block (observation window) of M samples and following the approach in [2], we can write equation (2) in vectorial notation as:

$$\mathbf{r}_i(\tau) = \mathbf{A}(\tau) \cdot \mathbf{x}_i + \mathbf{w}_i \quad (4)$$

where

$$\begin{aligned} \mathbf{x}_i &= \left[\begin{array}{ccc} c_i & \dots & c_{i+K-1} \end{array}\right]^T \\ \mathbf{a}_n &= \left[\begin{array}{cc} g(-nT-\tau) & \dots \end{array}\right. \\ &\qquad \left.\begin{array}{cc} \dots & g((M-1)T_s - nT - \tau) \end{array}\right]^T \\ \mathbf{A}(\tau) &= \left[\begin{array}{ccc} \mathbf{a}_0 & \dots & \mathbf{a}_{K-1} \end{array}\right]^T \\ \mathbf{w}_i &= \left[\begin{array}{ccc} w(iT) & \dots & w(iT + (M-1)T_s) \end{array}\right]^T \end{aligned} \quad (5)$$

and i is the block index, which will be omitted in subsequent sections.

3. COMPRESSED ML FORMULATION

The *ML* synchronizer selects the receiver timing τ' that minimizes the following *log-ML* cost function [2]:

$$\mathcal{C}(\tau, \tau', \mathbf{x}) = \|\mathbf{r}(\tau) - \mathbf{A}(\tau')\mathbf{x}\|^2 \quad (6)$$

This function also depends on the vector of transmitted pseudo-symbols $\mathbf{x}$, which is unknown at the receiver unless Data-Aided (*DA*) schemes are applied. Decision-Directed (*DD*) schemes replace $\mathbf{x}$ with the symbols decided at the decoder output but these decisions are not yet reliable in presence of timing errors. In *NDA* synchronizers the data nuisance can be removed in two different ways [2]:

1) *UML* synchronizers: they consider that vector $\mathbf{x}$ is random and compute the expectation of the (exponential)*ML* function with respect to $\mathbf{x}$.

2) *Compressed ML* synchronizers: they compress the *ML* function introducing into (6) one estimation of vector $\mathbf{x}$.

We will apply the second approach. The compressed *log-ML* cost function becomes then:

$$\mathcal{C}(\tau, \tau') = \|\mathbf{r}(\tau) - \mathbf{A}(\tau') \cdot \hat{\mathbf{x}}(\mathbf{r}(\tau), \tau')\|^2 \quad (7)$$

If we consider that the timing error $\varepsilon = \tau - \tau'$ is small (tracking condition), we can design $\hat{\mathbf{x}}(\mathbf{r}(\tau), \tau')$ to be optimal (under the criterion given in section 4) around $\tau' = \tau$ so that we can assume that $\hat{\mathbf{x}}(\mathbf{r}(\tau), \tau' \simeq \tau)$ is independent of the search parameter τ'. Thus only the cross terms of the square in (7) depend on τ' and (7) can be rewritten as follows:

$$\mathcal{C}(\tau, \tau') = \mathcal{F}(\tau) - 2 \cdot Re(\mathbf{r}(\tau)^H \mathbf{A}(\tau') \cdot \hat{\mathbf{x}}(\mathbf{r}(\tau), \tau' \simeq \tau)) \quad (8)$$

where $\mathcal{F}(\tau)$ includes all those terms which only depend on τ.

The derivative of (8) with respect to τ' has the following expression no matter what specific estimator $\hat{\mathbf{x}}(\mathbf{r}(\tau), \tau')$ is used:

$$\nabla \mathcal{C}(\tau, \tau') = -2 \cdot Re(\mathbf{r}(\tau)^H D(\tau') \hat{\mathbf{x}}(\mathbf{r}(\tau), \tau')) \quad (9)$$

where

$$D(\tau') = \frac{\partial}{\partial \tau'} A(\tau') \quad (10)$$

This gradient can be used to construct the following timing error discriminator:

$$\hat{\epsilon} = \mu \cdot Re\left(\mathbf{r}(\tau)^H D(\tau') \cdot \hat{\mathbf{x}}(\mathbf{r}(\tau), \tau')\right) \quad (11)$$

where μ is a constant selected to ensure the discriminator S-curve (expected value of $\hat{\epsilon}$) has unitary slope around $\tau = \tau'$ (no bias). That is,

$$\left. \frac{\partial E_{\mathbf{r}}(\hat{\varepsilon})}{\partial \tau} \right|_{\tau=\tau'} = 1 \quad (12)$$

The S-curves of the considered discriminators are plotted in section 5 (Figure 2).

4. ESTIMATION OF THE PSEUDO-SYMBOLS

The best linear estimator under a minimum mean-square error (MMSE) criterion was used in [2] [5] to derive the MCV-CML discriminator that attains the UCRBG (Gaussian Unconditional Cramer Rao Bound) in the whole *Eb/No* range. Its expression is:

$$\begin{aligned} &\hat{\mathbf{x}}(\mathbf{r}(\tau), \tau') = \mathbf{C}(\tau') \cdot \mathbf{r}(\tau) \\ &\mathbf{C}(\tau') = argmin_{\mathbf{C}(\tau')} E_{\mathbf{x},\mathbf{r}}(\|\mathbf{x} - \mathbf{C}(\tau') \cdot \mathbf{r}(\tau)\|^2) \\ &\mathbf{C}(\tau') = \mathbf{A}(\tau')^H (\mathbf{A}(\tau') \cdot \mathbf{A}(\tau')^H + \sigma_w^2 \mathbf{I})^{-1} \end{aligned} \quad (13)$$

It is well-known that this estimation would only be optimal if $\mathbf{x}$ was Gaussian. In general, the minimum mean-square error estimator is $\hat{\mathbf{x}} = E(\mathbf{x}/\mathbf{r})$ [7]. This estimator uses all the statistics of the data $\mathbf{x}$ whereas in (13) only the second order statistics are considered.

The fourth-order discriminator proposed by Mengali et al. ([1], section 9.4) showed that high-order discriminators variance could be below the *UCRBG* and close to the Cramer Rao Bound (CRB) [4]. This result prompted us to incorporate in (11) a third-order *MMSE* estimator of the pseudo-symbols $\hat{\mathbf{x}}$ (*MCV3*) with the aim of devising the optimal fourth-order discriminator that followed the equation (11). This discriminator will be named *MCV3-CML* hereafter.

The general expression of a third-order estimator can be stated as follows :

$$\hat{\mathbf{x}} = \mathbf{K}_0 + \mathbf{K}_1 \cdot \mathbf{r} + \mathbf{K}_2 \cdot \mathbf{R}_2 + \mathbf{K}_3 \cdot \mathbf{R}_3 \qquad (14)$$

In the last equation, dependencies with respect to τ and τ' have been omitted and non-linear terms have been introduced with the following notation:

$$\begin{aligned} \mathbf{R}_2 &= \mathbf{r}^* \otimes \mathbf{r} \\ \mathbf{R}_3 &= \mathbf{r} \otimes \mathbf{R}_2 = \mathbf{r} \otimes \mathbf{r}^* \otimes \mathbf{r} \end{aligned} \qquad (15)$$

where $\otimes$ stands for the Kronecker product of matrices.

Matrices $\mathbf{K}_0$, $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$ in (14) must be selected to minimize the mean-square error, that is:

$$\{\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3\} = argmin_{\ \mathbf{K}_0,\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3} E_{\mathbf{x},\mathbf{r}} \left(\|\mathbf{x} - \hat{\mathbf{x}}\|^2 \right) \qquad (16)$$

It can be shown that the solution to (16) are these two equation systems:

$$\begin{array}{lclcl} \mathbf{K}_0 & + & \mathbf{K}_2 \cdot \bar{\mathbf{R}}_2 & = & \mathbf{0} \\ \mathbf{K}_0 \cdot \bar{\mathbf{R}}_2^H & + & \mathbf{K}_2 \cdot \bar{\mathbf{R}}_{4,2} & = & \mathbf{0} \end{array} \qquad (17)$$

$$\begin{array}{lclcl} \mathbf{K}_1 \cdot \bar{\mathbf{R}}_{2,1} & + & \mathbf{K}_3 \cdot \bar{\mathbf{R}}_{4,1} & = & \bar{\mathbf{R}}_{xr} \\ \mathbf{K}_1 \cdot \bar{\mathbf{R}}_{4,3} & + & \mathbf{K}_3 \cdot \bar{\mathbf{R}}_{6,3} & = & \bar{\mathbf{R}}_{xR_3} \end{array} \qquad (18)$$

in which the following notation has been used:

$$\begin{aligned} R_n &= \begin{cases} r \otimes R_{n-1} & \text{if n is odd} \\ r^* \otimes R_{n-1} & \text{if n is even} \end{cases} \\ \bar{R}_n &= E(R_n) \\ R_{n,m} &= R_{n-m} R_m^H \qquad (n > m > 0) \\ \bar{R}_{n,m} &= E(R_{n,m}) \\ \bar{R}_{xr} &= E(x r^H) \\ \bar{R}_{xR_3} &= E(x R_3^H) \end{aligned} \qquad (19)$$

where by definition $\mathbf{R}_1 \equiv \mathbf{r}$

Both equation systems, (17) and (18), are underdetermined, i.e., have infinite solutions, because matrices $\mathbf{R}_{4,m}$ and $\mathbf{R}_{6,m}$ have repeated elements. For example, the next three elements of matrices $\mathbf{R}_{4,m}$ are in fact the same moment:

$$E(r_i \cdot r_j^* \cdot r_k \cdot r_l^*) = E(r_k \cdot r_l^* \cdot r_i \cdot r_j^*) = E(r_i \cdot r_l^* \cdot r_k \cdot r_j^*) \qquad (20)$$

The minimum norm solution can be deduced by using the singular values decomposition (*SVD*) [7]. Notice also that

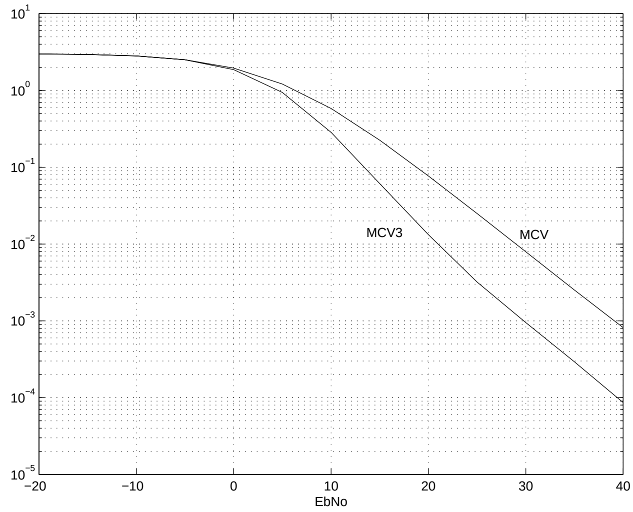

Figure 1: Mean-square error of the estimated symbols for *MCV* and *MCV3*

$\mathbf{K}_0 = \mathbf{0}$ and $\mathbf{K}_2 = \mathbf{0}$ are the minimum norm solutions for $\mathbf{K}_0$ and $\mathbf{K}_2$ in (17).

Matrices $\mathbf{R}_{n,m}$, $\mathbf{R}_{xr}$ and $\mathbf{R}_{xR_3}$ are dependent on the modulation and can be computed easily when data windows are not very large. In Figure 1 *MCV* and *MCV3* mean-square error is compared for a window of size M=4. Their performance is not good because of the short window, but the improvement of the third order estimator is appreciable, mainly as *Eb/No* increases.

5. SIMULATION RESULTS

In this section the fourth order closed-loop timing discriminator presented in this paper (*MVC3-CML*) is simulated and compared with the *DA* (Data-Aided) scheme and the *MCV-CML* discriminator in terms of tracking variance for the *AWGN* channel and the *MSK* modulation described in Section 2.

The discriminator has been designed for a small window size (M=4) and a sampling rate equal to $2/T$. Despite the small window size, good results are achieved for medium to high *SNRs* as depicted in Figure 3.

Several important conclusions can be drawn from Figure 3:

1) For medium to high *SNRs*, the *MCV3-CML* variance is lower than that one of *MVC-CML* because the third-order term in (14) improves the symbols estimation without appreciable noise enhancement;

2) It has been verified that the fourth-order term in (11) remains significant as σ_w^2 increases because linear estimators can not comply with the second equation in (18);

3) For low *SNRs*, *MCV3-CML* variance tends toward that one of the *MCV-CML* quadratic solution and even becomes slightly higher at very low *SNRs* ($\approx 0.77dB$). This seems

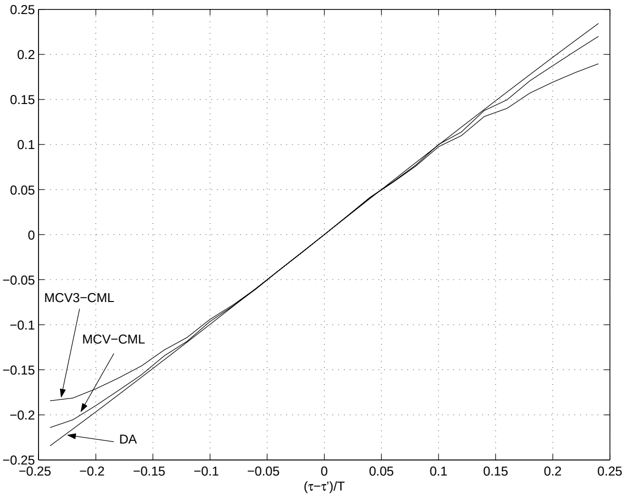

Figure 2: S-curve for DA, MCV-CML and MCV3-CML. In all the cases the sampling rate is $2/T$.

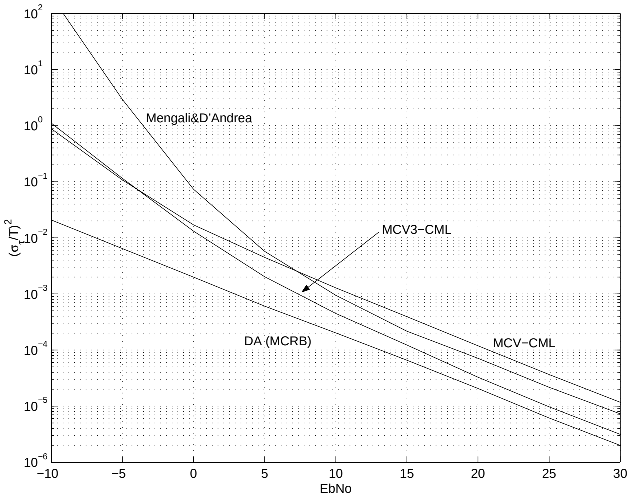

Figure 3: Normalized timing variance (σ_τ^2/T^2) for the studied discriminators: DA, MCV-CML, MCV3-CML and the Mengali and D'Andrea discriminator ([1], Section 9.4). $B_N \cdot T = 5 \cdot 10^{-3}$

reasonable because, under the assumption that the symbols are unknown, the *ML* estimator becomes asymptotically quadratic at low *SNR* [2];
4) Both *MCV3-CML* and *MCV-CML* variance grows proportionally to σ_w^2 at high *SNRs* and to σ_w^4 at low *SNRs*.

In Figure 2 the S-curves of the three discriminators are plotted. It can be easily proved that the slope of the S-curve for the *MCV-CML* and *MCV3-CML* algorithms depends on the noise variance σ_w^2 and hence the parameter μ in (11). Figure 2 also shows that the *MCV3-CML* acquisition behaviour is a little worse.

6. CONCLUSIONS

In this paper a new general approach for the design of fourth order *NDA* timing synchronizers has been presented. Notice that this approach can also be applied to frequency estimation and to any other parameter embedded in a communication signal. The methodology proposed here is attractive because the discriminator structure does not depend on the modulation scheme. The only term dependent on the modulation is the estimator of the (pseudo-)symbols $\hat{\mathbf{x}}$ that can be designed separately.

The core of the paper is devoted to improving the decorrelation of the (pseudo-)symbols by using a statistical third order estimator. This estimator gains benefit from the fourth and sixth order statistics of the received data which are function of the noise variance and the symbols distribution. Simulations show that the fourth order discriminator behaviour is better at high *SNRs* and converges to the quadratic discriminator at low *SNRs*.

7. REFERENCES

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