A WINDOWING CONDITION FOR CHARACTERIZATION OF FINITE SIGNALS FROM SPECTRAL PHASE OR MAGNITUDE

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ABSTRACT

Reconstruction of a signal from its spectral phase or magnitude is in general an ill-posed problem. Various conditions restricting the class of signals under consideration have been shown to be sufficient to regularize the problem so that a unique (or essentially unique) signal corresponds to any given spectral magnitude or spectral phase function. This paper shows that a finite discrete-time signal is characterized by its spectral magnitude (or phase) and the spectral magnitude (or phase) of an ancillary signal obtained by windowing the original signal.

1. INTRODUCTION

The problem of recovering or uniquely defining a signal from its Fourier (spectral) phase or magnitude has attracted ongoing attention over several decades in connection with signal processing, crystallography, optics, and a few other application areas. These are commonly referred to in research literature as the phase and magnitude retrieval problems, respectively. In both continuous-time and discrete-time settings, neither the magnitude nor phase of the Fourier transform is sufficient, in general, to characterize a signal. Consequently, both the phase and magnitude retrieval problems fall into the class of ill-posed inverse problems.

Over the years, various restrictions on the class of signals considered have been shown to be sufficient to regularize the phase or magnitude retrieval problem so that a unique (or essentially unique) signal within the restricted class corresponds to any given spectral phase or magnitude function. In particular, for finite discrete-time signals, minimum-phase and maximum-phase signals are uniquely determined to within a constant by their spectral phase or magnitude. Hayes [1] extended this type of result to a class of signals whose z-transform zeros do not occur in conjugate reciprocal pairs. Among numerous other results in the literature (see, for example, the papers [2, 3, 5, 6] and the books [4, 10]), the authors have recently proven uniquness theorems for real-valued finite discrete-time signals using knowledge of a few strategically placed samples [7, 8].

This paper shows that finite discrete-time signals are uniquely determined up to constant factors by the magnitude of their (discrete-time) Fourier transform and the magnitude of the Fourier transform of an ancillary signal obtained by windowing the original signal. A similar result is obtained for spectral phase. The paper closes with an open question about the generalization of the results presented.

2. NOTATION AND PRELIMINARIES

Denote by $x[n]$ a one-dimensional sequence supported on the interval $0 \leq n \leq N-1$ and define polynomials

$$X(z) = \sum_{n=0}^{N-1} x[n] z^n \tag{1}$$

$$X_*(z) = z^{N-1}\overline{X(\frac{1}{\overline{z}})} = \sum_{n=0}^{N-1} \overline{x[n]} z^{N-1-n} \tag{2}$$

where the bar denotes complex conjugation. $X(z)$ is often referred to as the z-transform of $x[n]$, though the engineering literature normally would define the z-transform of such a signal as a polynomial in z^{-1}. For the purposes of this paper, the notation (1) is convenient, though the results can be obtained using either definition. The discrete-time Fourier transform (DTFT) of $x[n]$ is obtained by evaluating $X(z)$ on unit circle $|z| = 1$; i.e., $\hat{X}(\omega) = X(e^{i\omega})$ for all real ω. The DTFT can be represented in terms of its spectral magnitude function $|\hat{X}(\omega)|$ and spectral phase function $\phi(\omega)$ by

$$\hat{X}[\omega] = |\hat{X}(\omega)| e^{i\phi(\omega)} \tag{3}$$

To ensure that $\phi(\omega)$ is well defined it will be assumed that $X(z)$ has no zeros on the unit circle. The following lemma, which is straightforward to prove, gives conditions under which two signals of the type being considered have identical spectral magnitude functions or spectral phase functions. These conditions are used in the following section to establish the main results of this paper.

Lemma 1 *Consider two discrete-time signals $x[n]$ and $y[n]$, both supported on the interval $0 \leq n \leq N-1$. These signals have the same spectral magnitude function if and only if $X(z)X_*(z) = Y(z)Y_*(z)$ (or $|X(z)| = |Y(z)|$ on $|z|=1$). They have the same spectral phase function if and only if $X(z)Y_*(z) = Y(z)X_*(z)$ (or $\angle X(z) = \angle Y(z)$ on $|z|=1$).*

3. UNIQUENESS THEOREMS

For a given signal $x[n]$ of the type under consideration, there is a unique "ancillary" signal obtained as the inverse z-transform of the derivative $X'(z)$ of $X(z)$. The first key result of this section shows that $x[n]$ is determined up to a unimodular constant factor by knowledge of *both* $|X(z)|$ and $|X'(z)|$ on the unit circle (i.e., by the spectral magnitude functions of the original signal *and* the ancillary signal). The second result shows that a singal is determined up to a positive constant factor by its spectral phase function *and* the spectral phase function the ancillary signal. The development proceeds as a sequence of lemmas. The proofs of some of these are omitted for brevity, but they can ve verified by direct calculation starting with the appropriate definitions.

3.1. Characterization from spectral magnitude

Lemma 2 *Let $X(z)$ be a polynomial of degree $N-1$ representing the z-transform of a finite signal of length N). Then $zX_*'(z) = (N-1)X_*(z) - (X')_*(z)$.*

Proof: Recall that

$$X_*(z) = z^{N-1}\overline{X(\frac{1}{\bar{z}})} = \sum_{j=0}^{N-1} \overline{x[j]} z^{N-1-j}$$

Thus

$$X_*'(z) = \sum_{j=0}^{N-1} (N-1-j)\overline{x[j]} z^{N-j-2} \tag{4}$$

and hence, denoting $M = N-1$,

$$\begin{aligned} zX_*'(z) &= M\sum_{j=0}^{M} \overline{x[j]} z^{M-j} - \sum_{j=0}^{M} j\overline{x[j]} z^{(M-1)-(j-1)} \\ &= MX_*(z) - z^{k-1}\sum_{j=0}^{M} j\overline{x[j]}(\frac{1}{z})^{j-1} \\ &= (N-1)X_*(z) - (X')_*(z) \end{aligned}$$

Lemma 3 *(Logarithmic Derivative) If $X(z)$ and $Y(z)$ are the z-transforms of two finite length non-zero signals then*

$$\frac{(X(z)Y(z))'}{X(z)Y(z)} = \frac{X'(z)}{X(z)} + \frac{Y'(z)}{Y(z)}$$

Proof: Direct calculation.

Lemma 4 *If $|X(z)| = |Y(z)|$ on $|z|=1$, then there exist polynomials $U(z)$ and $V(z)$ such that $X(z) = U(z)V(z)$ and $Y(z) = cU(z)V_*(z)$ for some constant c.*

Lemma 5 *For polynomials $U(z)$ and $V(z)$, $(UV)_*(z) = U_*(z)V_*(z)$.*

Theorem 6 *If $|X(z)| = |Y(z)|$ and $|X'(z) = |Y'(z)|$ on $|z|=1$, then $X(z) = cY(z)$ for some complex constant c of unit modulus.*

Proof: Since $|X(z)| = |Y(z)|$ on $|z|=1$, lemma 4 gives

$$\begin{aligned} X(z) &= U(z)V(z) \\ Y(z) &= U(z)V_*(z) \end{aligned} \tag{5}$$

From lemma 3,

$$\begin{aligned} \frac{X'(z)}{X(z)} &= \frac{U'(z)}{U(z)} + \frac{V'(z)}{V(z)} \\ \frac{Y'(z)}{Y(z)} &= \frac{U'(z)}{U(z)} + \frac{V_*'(z)}{V_*(z)} \end{aligned} \tag{6}$$

From lemma 5,

$$X'(z)(X')_*(z) = Y'(z)(Y')_*(z)$$

Denoting $M = N-1$ and using lemma 2 this becomes

$$X'(z)(MX_*(z) - zX_*'(z)) = Y'(z)(MY_*(z) - zY_*'(z))$$

Dividing both sides by $X(z)X_*(z) = Y(z)Y_*(z)$ gives

$$\frac{X'(z)}{X(z)}((N-1) - z\frac{X_*'(z)}{X_*(z)}) = \frac{Y'(z)}{Y(z)}((N-1) - z\frac{Y_*'(z)}{Y_*(z)}) \tag{7}$$

By lemma 5 and equation (5), $X_*(z) = U_*(z)V_*(z)$ and $Y_*(z) = U_*(z)V(z)$. Thus, by lemma 3,

$$\begin{aligned} \frac{X_*'(z)}{X_*(z)} &= \frac{U_*'(z)}{U_*(z)} + \frac{V_*'(z)}{V_*(z)} \\ \frac{Y_*'(z)}{Y_*(z)} &= \frac{U_*'(z)}{U_*(z)} + \frac{V'(z)}{V(z)} \end{aligned} \tag{8}$$

Substituting equations (6) and (8) into (7) and simplifying yields

$$\Big(\frac{V'(z)}{V(z)} - \frac{V_*'(z)}{V_*(z)}\Big)\Big(M - z\Big(-\frac{U'(z)}{U(z)} + \frac{U_*'(z)}{U_*(z)}\Big)\Big) = 0$$

It follows that either

$$\Big(\frac{V^{'}(z)}{V(z)}-\frac{V_*^{'}(z)}{V_*(z)}\Big)=0 \tag{9}$$

or

$$(N-1)=z\Big(-\frac{U^{'}(z)}{U(z)}+\frac{U_*^{'}(z)}{U_*(z)}\Big) \tag{10}$$

If (9) holds, then $V_*(z)V'(z)-V_*'(z)V(z)=0$, which implies

$$\Big(\frac{V(z)}{V_*}\Big)^{'}=0$$

and thus $V(z)=cV_*(z)$ for some constant c. Hence, in this case, $X(z)=cY(z)$ where the modulus of c is clearly one. Alternatively, suppose equation (10) holds. Then $U(z)=0$ follows from expansion of $\Big(z^{(N-1)}\frac{U(z)}{U_*(z)}\Big)^{'}$. Hence $z^{(N-1)}U(z)=cU_*(z)$. Suppose $U(z)$ has degree $k\geq 0$. Since $N-1\geq k$,

$$z^{(N-1)}U(z)=cz^k\overline{U(\frac{1}{\bar{z}})}$$

which implies

$$z^{N-1-k}U(z)=c\overline{U(\frac{1}{\bar{z}})}$$

But since the RHS is a polynomial in $\frac{1}{z}$, the LHS is a polynomial in z. It follows that $N-1=k$ and hence $X(z)=cY(z)$.

3.2. Characterization from spectral phase

An equivalent version of theorem 6 is developed in the context of phase information which shows that a signal $x[n]$ can be uniquely defined by its spectral phase and the spectral phase of the signal $y[n]$ where $y[n]$ is as defined in equation 17.

Lemma 7 *If $X(z)$ and $Y(z)$ are polynomials of degree $N-1$ having no zeros on the unit circle and $\angle X(z)=\angle Y(z)$ on $|z|=1$, then there exist polynomials $U(z)$, $V_1(z)$, and $V_2(z)$ such that*

$$X(z)=U(z)V_1(z),\ \ Y(z)=U(z)V_2(z)$$

and

$$V_1(z)=(V_1)_*(z),\ \ V_2(z)=(V_2)_*(z)$$

Theorem 8 *If $X(z)$ and $Y(z)$ are as in the preceding lemma,*

$$\frac{V_1'(z)}{V_1(z)}-\frac{V_2'(z)}{V_2(z)}=\frac{(V_{1*})^{'}(z)}{V_{1*}(z)}-\frac{(V_{2*})^{'}(z)}{V_{2*}(z)}$$

where $V_1(z)$ and $V_2(z)$ are as described in lemma 7.

Theorem 9 *If $X(z)$ and $Y(z)$ are as in lemma 7, and $\angle X^{'}(z)=\angle Y^{'}(z)$ on $|z|=1$ then $X(z)=cY(z)$ for some constant c.*

Proof: *CASE A: No common zeros.* If $X(z)$ and $Y(z)$ have no common zeros, then from lemma 7 $X(z)=X_*(z)$ and $Y(z)=Y_*(z)$. From lemmas 1 and 2 (and denoting $M=N-1$)

$$X^{'}(z)\Big(MY_*(z)-zY_*^{'}(z)\Big)=Y^{'}(z)\Big(MX_*(z)-zX_*^{'}(z)\Big) \tag{11}$$

Dividing both sides of the equation by $X_*(z)Y(z)=Y_*(z)X(z)$ gives

$$\frac{X^{'}(z)}{X(z)}\Big(M-z\frac{Y_*^{'}(z)}{Y_*(z)}\Big)=\frac{Y^{'}(z)}{Y(z)}\Big(M-z\frac{X_*^{'}(z)}{X_*(z)}\Big)$$

so that

$$M\Big(\frac{X^{'}(z)}{X(z)}-\frac{Y^{'}(z)}{Y(z)}\Big)=z\Big(\frac{X^{'}(z)Y_*^{'}(z)}{X(z)Y_*(z)}-\frac{Y^{'}(z)X_*^{'}(z)}{Y(z)X_*(z)}\Big)$$

and

$$M\Big(\frac{X^{'}(z)}{X(z)}-\frac{Y^{'}(z)}{Y(z)}\Big)=0$$

The LHS reduces to zero since $X(z)=X_*(z)$ and $Y(z)=Y_*(z)$. Hence $X^{'}(z)Y(z)-X(z)Y^{'}(z)=0$ so that $(X(z)/Y(z))'=0$ and thus $X(z)=cY(z)$.

CASE B: Common zeros present. From lemma 7, $X(z)$ and $Y(z)$ may be written $X(z)=U(z)V_1(z)$ and $Y(z)=U(z)V_2(z)$ where $V_1(z)=(V_1)_*(z)$ and $V_2(z)=(V_2)_*(z)$. Lemma 3 implies

$$\begin{aligned}\frac{X^{'}(z)}{X(z)}&=\frac{U^{'}(z)}{U(z)}+\frac{V_1^{'}(z)}{V_1(z)}\\ \frac{Y^{'}(z)}{Y(z)}&=\frac{U^{'}(z)}{U(z)}+\frac{V_2^{'}(z)}{V_2(z)}\end{aligned} \tag{12}$$

From lemmas 1 and 2,

$$X^{'}(z)\Big(MY_*(z)-zY_*^{'}(z)\Big)=Y^{'}(z)\Big(MX_*(z)-zX_*^{'}(z)\Big)$$

Dividing by $X_*(z)Y(z)=Y_*(z)X(z)$ gives

$$\frac{X^{'}(z)}{X(z)}\Big(M-z\frac{Y_*^{'}(z)}{Y_*(z)}\Big)=\frac{Y^{'}(z)}{Y(z)}\Big(M-z\frac{X_*^{'}(z)}{X_*(z)}\Big) \tag{13}$$

Substituting equation (12) into equation (13) and simplifying yields

$$M\Big(\frac{V_1^{'}(z)}{V_1(z)}-\frac{V_2^{'}(z)}{V_2(z)}\Big)-z\Big(\frac{V_1^{'}(z)}{V_1(z)}-\frac{V_2^{'}(z)}{V_2(z)}\Big)\Big(\frac{U_*^{'}(z)}{U_*(z)}-\frac{U^{'}(z)}{U(z)}\Big)$$

$$=z\Big(\frac{V_2^{'}(z)V_{1*}^{'}(z)}{V_2(z)V_{1*}(z)}-\frac{V_1^{'}(z)V_{2*}^{'}(z)}{V_1(z)V_{2*}(z)}\Big)$$

Since $V_1(z) = V_{1*}(z)$ and $V_2(z) = V_{2*}(z)$ the LHS reduces to zero and

$$\Big(\frac{V_1'(z)}{V_1(z)} - \frac{V_2'(z)}{V_2(z)}\Big)\Big(M - z\Big(\frac{U_*'(z)}{U_*(z)} - \frac{U'(z)}{U(z)}\Big)\Big) = 0 \quad (14)$$

This implies that either

$$\frac{V_1'(z)}{V_1(z)} - \frac{V_2'(z)}{V_2(z)} = 0 \quad (15)$$

or

$$M = z\Big(\frac{U_*'(z)}{U_*(z)} - \frac{U'(z)}{U(z)}\Big) \quad (16)$$

From equation (15),

$$\Big(\frac{V_1(z)}{V_2(z)}\Big)' = 0$$

which implies $V_1(z) = cV_2(z)$ for some constant c. The condition arising from equation (16) is similar to the condition from equation (10) and hence leads to the conclusion that $U(z)$ is constant. Hence there are in fact no non-common zeros and this reduces to *CASE A*.

3.3. A windowing perspective

Given a signal $x[n]$ of the type under discussion, consider the signal $y[n]$ defined as $y[n] = w[n]x[n]$ where $w[n]$ is some windowing signal. One might ask whether, for suitable $w[n]$, knowledge of the spectral magnitude of both the original signal $x[n]$ and the windowed signal $y[n]$ is sufficient to determing $x[n]$. Similary, is knowledge of the spectral phase of both $x[n]$ and $y[n]$ ever sufficient to determine $x[n]$?

The above results show that, for the particular window defined by $w[n] = n$ for $0 \leq n \leq N-1$ and $w[n] = 0$ otherwise, the answer to both questions is "yes" up to a multiplicative constant. On the other hand, if $w[n]$ is a constant window (i.e., $w[n] = 1$ for all n), the the answer is clearly "no." This perspective raises the question of what windows are suitable for extraction of spectral phase or spectral magnitude information from a signal in this fashion.

4. CONCLUDING REMARKS

It has been shown in this paper that a finite discrete-time signal is essentially uniquely determined by either its spectral phase and the spectral phase of an ancillary signal defined through differentiation of the original signal's z-transform. A similar result was shown for spectral magnitude. Future research will examine the open question regarding windowing raised above.

5. REFERENCES

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