EFFICIENT QUANTIZATION FOR OVERCOMPLETE EXPANSIONS IN $\mathcal{R}^N$

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ABSTRACT

The use of quantized redundant expansions is useful in applications where the cost of having oversampling in the representation is much lower than the use of a high resolution quantization (e.g. oversampled A/D). Most work to date has assumed that simple uniform quantization was used on the redundant expansion and then has dealt with methods to improve the reconstruction. Instead, in this paper we consider the design of quantizers for overcomplete expansions. Our goal is to design quantizers such that simple reconstruction algorithms (e.g. linear) provide as good reconstructions as with more complex algorithms. We achieve this goal by designing quantizers with different stepsizes for each coefficient of the expansion in such a way as to produce a quantizer with periodic structure.

1. INTRODUCTION AND MOTIVATION

The purpose of using redundant expansions is to achieve high accuracy in digital signal representations under scenarios where the cost of implementing a high resolution quantization with the current technology is much higher than that of having a high oversampling or redundancy. The most important practical case is that of oversampled A/D conversion of band-limited signals, where accuracy is attained by performing an oversampling in the time domain.

The accuracy that can be attained with quantized overcomplete expansions depends on two things: the reconstruction algorithm and the quantization scheme. Unlike previous work [1, 6, 5, 3], which assumes a known quantization and focuses on improving the reconstruction, in our work we assume that a simple reconstruction (e.g. linear or look-up table) will be used. Our approach has focused on providing the tools to design the overcomplete expansions and corresponding quantization system so that the equivalent quantizer is regular under simple reconstruction algorithms. We restrict ourselves to using scalar quantizers for each component of the expansion, but allow the stepsizes to be different in each component. The approaches developed previously in [3, 1] can be applied in $\mathcal{R}^N$ for any finite values of N and r, while the method described in [5] can be applied to oversampled A/D of general band-limited signals (in addition to periodic band-limited signals, which can be viewed as a particular quantized frame expansion in $\mathcal{R}^N$).

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In all cases, the computational complexity of these methods, for a given redundancy, is higher than that of linear reconstruction while they provide substantially better accuracy for high enough redundancies. Our system provides excellent performance while having the same complexity as linear reconstruction, but is more suitable to be used in $\mathcal{R}^N$ for low values of dimension and redundancy.

This paper is organized as follows. First in section 2 we describe the oversampled representation system in terms of an equivalent VQ system. This allows us to introduce in section 3 the concept of periodic quantizer that leads to an equivalent regular VQ scheme. Numerical results are shown in section 4.

2. LINEAR RECONSTRUCTION, CONSISTENCY AND EQUIVALENT VQ

Let $\boldsymbol{x} \in \mathcal{R}^N$ and let $\Phi = \{\boldsymbol{\varphi}_i\}_{i=1}^M$ be a tight frame in $\mathcal{R}^N$ with $\|\boldsymbol{\varphi}_i\| = 1 \; \forall \; i = 1, \cdots, M$. Then, $\forall \; \boldsymbol{x} \in \mathcal{R}^N$, the expansion with respect to the frame $\Phi = \{\boldsymbol{\varphi}_i\}_{i=1}^M$ whose coefficients have the minimum possible norm is given by [6]:

$$\boldsymbol{x} = \frac{1}{r}\sum_{i=1}^{M} y_i \boldsymbol{\varphi}_i = \frac{1}{r}\sum_{i=1}^{M} \langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle \boldsymbol{\varphi}_i \qquad (1)$$

where $y_i = \langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle$ is the i-th coefficient of the frame. We restrict our attention in this paper to tight frames with integer redundancy r and composed of a set of orthogonal bases. This is both because of the greater simplicity of the geometric analysis and due to their relevance for practical applications. With this restriction, we can group the vectors $\{\boldsymbol{\varphi}_i\}_{i=1}^M$ that compose the tight frame as $\{\{\boldsymbol{\varphi}_i^j\}_{i=1}^N\}_{j=1}^r$, where $\{\boldsymbol{\varphi}_i^j\}_{i=1}^N$ is the j-th basis. The i-th coefficient of the j-th basis is quantized with a uniform quantizer with stepsize Δ_i^j. In general all these quantizers are assumed to be different. For the sake of simplicity, we restrict most of the equations, without any loss of generality, to $\mathcal{R}^2$. For $N = 2$, we define each orthogonal matrix $\boldsymbol{F}^j$ as $\boldsymbol{F}^j = [\boldsymbol{\varphi}_1^j \boldsymbol{\varphi}_2^j]^T$ and we call $\boldsymbol{y}^j = [y_1^j, y_2^j]^T$ the 2-dimensional vector of coefficients associated with the j-th basis, which is given by $\boldsymbol{y}^j = \boldsymbol{F}^j \boldsymbol{x}$.

In terms of quantization we have a series of orthogonal basis over which a scalar quantizer is used. Considering in isolation each of these structures lead to a partition of $\mathcal{R}^2$ with a rectangular grid, i.e., with rectangular (or hypercubic) Voronoi regions. Clearly, each of these grids is defined in terms of a lattice where the quantization stepsizes determine the position of the vertices. The corresponding real lattice Λ^j has a generator matrix $\boldsymbol{M}_{\Lambda^j} = (\Delta_1^j \boldsymbol{\varphi}_1^j | \Delta_2^j \boldsymbol{\varphi}_2^j)^T$.

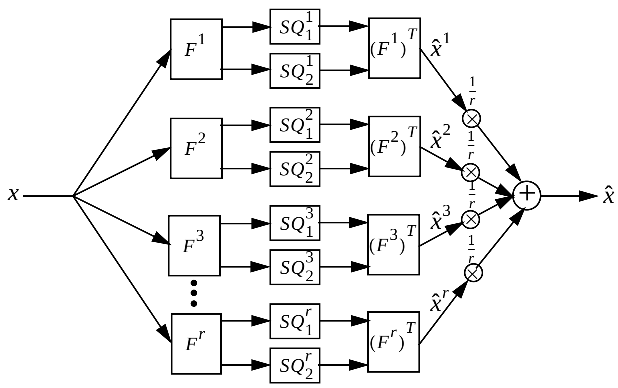

Fig. 1. Definition of a quantizer Q in $\mathcal{R}^2$ based on the linear reconstruction of a tight frame

Now consider quantization using several of these orthogonal bases jointly in an overcomplete system. Clearly, the resulting Voronoi regions of the combined system are the intersections of the Voronoi regions of the individual grids. This is illustrated in Figs. 1 and 2.

Linear reconstruction is illustrated in Fig. 2. An input that falls in the intersection of two specific (rectangular) Voronoi regions is reconstructed as the average of the centers of those two regions. In some cases, as for the upper cell (indicated with bold line), the average point falls again in the same intersection so that we have a regular quantization and consistent reconstruction. However, in the case of the lower cell, the average point falls outside the original region of intersection thus leading to inconsistency.

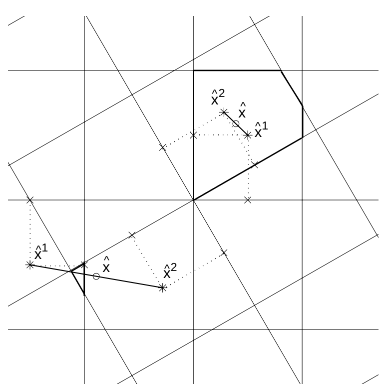

Fig. 2. Reconstructions for the quantizers Q^1, Q^2 and Q when linear reconstruction is used. The linear reconstructions $\hat{\boldsymbol{x}}^j$, $j = 1, 2$ are represented by $'*'$ and the final reconstruction $\hat{\boldsymbol{x}}$ is represented by $'\circ'$. The final reconstructions are obtained by taking the halfway point between $\hat{\boldsymbol{x}}^1$ and $\hat{\boldsymbol{x}}^2$, that is, $\hat{\boldsymbol{x}} = \frac{1}{2}(\hat{\boldsymbol{x}}^1 + \hat{\boldsymbol{x}}^2)$.

3. QUANTIZERS WITH PERIODIC STRUCTURE

3.1. Definition and Construction

A "periodic quantizer" Q is a quantizer where there is only a finite number of distinct Voronoi cells $\{V_i^Q\}$ (see Fig. 3). Let us assume to facilitate the understing that Q is a quantizer in $\mathcal{R}^2$ and $r = 2$.

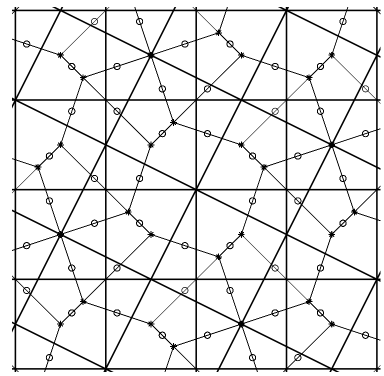

Fig. 3. Example of a linearly consistent quantizer Q. The reconstructions $\hat{\boldsymbol{x}}^j$, $j = 1, 2$ are represented by $'*'$ and the final reconstruction $\hat{\boldsymbol{x}}$ is represented by $'\circ'$. Parameters: $\Delta_2^1 = \Delta_1^1$, $\Delta_2^2 = \Delta_1^2 = \frac{\sqrt{5}}{2}\Delta_1^1$, $\tan(\theta) = \frac{1}{2}$

In order to impose a periodic structure in Q, we have to find certain type of sublattices [4]. A sublattice $S\Lambda \subset \Lambda$ of a given lattice Λ is a subset of the elements of Λ that is itself a lattice. Given a real lattice Λ with generator matrix $\boldsymbol{M}_\Lambda$, a sublattice[1] $S\Lambda$ is completely specified by an integer matrix $\boldsymbol{B}_{S\Lambda}$ that maps a basis of Λ into a basis of $S\Lambda$, that is, $\boldsymbol{M}_{S\Lambda} = \boldsymbol{B}_{S\Lambda}\boldsymbol{M}_\Lambda$.

Definition 1 *Given a real lattice Λ in $\mathcal{R}^2$ with generator matrix $\boldsymbol{M}_\Lambda$, a lattice Λ' is geometrically scaled-similar to Λ iff:*

$$\boldsymbol{M}_{\Lambda'} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \boldsymbol{U}\boldsymbol{M}_\Lambda\boldsymbol{R}, \quad c_1, c_2 \geq 1 \qquad (2)$$

where $\boldsymbol{R}$ is a 2×2 orthogonal matrix, that is, a rotation and/or a reflection in $\mathcal{R}^2$, and $\boldsymbol{U}$ is a 2×2 unimodular matrix, that is, a matrix with integer components satisfying that $|det(\boldsymbol{U})| = 1$. If $\Lambda' = S\Lambda \subset \Lambda$, then $S\Lambda$ is a geometrically scaled-similar sublattice of Λ and c_1, c_2 and $\boldsymbol{R}$ are constrained.

It can be seen in (2) that a sublattice $S\Lambda$ is obtained by simply rotating and/or reflecting the lattice Λ and then scaling each of the new rotated axes by a certain amount. Notice that in the particular case of having $c_1 = c_2$, $S\Lambda$ would be a geometrically similar (or equivalent) sublattice of Λ, as defined by Conway et al. [4]. Fig. 4 shows an example of sublattice for $r = 2$, where the cell indicated with bold line is the fundamental polytope of $S\Lambda$, which we denote by $V_o^{S\Lambda}$. Without loss of generality, let us assume we want to construct geometrically scaled-similar sublattices of a lattice Λ^1 where $\boldsymbol{M}_{\Lambda^1} = diag[\Delta_1^1, \Delta_2^1]$, which defines a quantizer Q^1. If there are r sublattices of Λ^1, we will denote them by $S\Lambda^1$, $S\Lambda^2$, ..., $S\Lambda^r$, and for notational convenience, we take $S\Lambda^1 = \Lambda^1$. We will always take $U = I$ in (2) so that the basis vectors of the j-th geometrically scaled-similar sublattice are orthogonal and can be associated with the j-th orthogonal basis of a tight frame.

Given Λ^1, it can be shown [2] that if $S\Lambda$ is a geometrically scaled-similar sublattice of Λ^1, the partition defined by $\{V_i^{\Lambda^1}\} \cap \{V_i^{S\Lambda}\}$ has a periodic structure (tesselation) with the basic unit cell being $\{V_i^{\Lambda^1}\} \cap V_o^{S\Lambda}$ (see Fig. 4). The following Lemma can be

[1] We assume that both Λ and $S\Lambda$ are full rank lattices, that is, the matrices $\boldsymbol{M}_\Lambda$ and $\boldsymbol{M}_{S\Lambda}$ are full rank

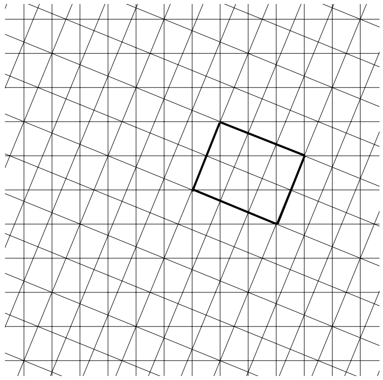

Fig. 4. Example 1: Voronoi cells $\{V_i^Q\}$. The sublattice structure is indicated with bold line. Parameters: $\beta = \sqrt{\frac{3}{2}}$, $\Delta_2^1 = \beta\Delta_1^1$, $\Delta_1^2 = \frac{1}{2\cos(\theta)}\Delta_1^1$, $\Delta_2^2 = \frac{1}{3\cos(\theta)}\beta\Delta_1^1$, $\tan(\theta) = \sqrt{6}$

proved in a straightforward manner by using the definition of geometrically scaled-similar sublattices.

Lemma 1 *Given a rectangular lattice Λ^1 in $\mathcal{R}^2$ whose generator matrix is diagonal, all the geometrically scaled-similar sublattices $S\Lambda \subset \Lambda^1$ (with the matrix $\boldsymbol{R}$ in (2) constrained to be a rotation), have generator matrices of the form:*

$$\begin{aligned}\boldsymbol{M}_{S\Lambda} &= \begin{pmatrix} c_1\Delta_1^1 & 0 \\ 0 & c_2\beta\Delta_1^1 \end{pmatrix}\begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} k_{11} & k_{12} \\ -k_{21} & k_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}\Delta_1^1 \qquad \textit{where}\end{aligned}$$

$$\begin{gathered}\beta = \frac{\Delta_2^1}{\Delta_1^1} = \sqrt{\frac{k_{11}k_{21}}{k_{12}k_{22}}}, \quad \tan(\theta) = \sqrt{\frac{k_{12}k_{21}}{k_{11}k_{22}}} = \frac{k_{12}}{k_{11}}\beta \\ c_1 = \frac{k_{11}}{cos(\theta)}, \quad c_2 = \frac{k_{22}}{cos(\theta)}, \\ k_{11}, k_{12}, k_{21}, k_{22} \in \mathcal{Z}_+, \quad 0 < \theta < \frac{\pi}{2}\end{gathered}$$

We denote by $\boldsymbol{B}_{S\Lambda}$ the integer matrix with entries $\{k_{lm}\}$.

Notice that only those angles θ, whose tangent is the square root of two integers lead to a geometrically scaled-similar sublattice.

In order to construct a general periodic quantizer Q for a redundancy r, it is sufficient to design the system so that there are $r-1$ lattices $\Lambda^2, \cdots, \Lambda^r$ each containing a sublattice of Λ^1. This guarantees that the intersection of all the lattices Λ^j, $j = 1, \cdots, r$ is not empty, and therefore, by group theory, is a lattice. In general, the lattices $\Lambda^j \supset S\Lambda^j$ ($S\Lambda^j$ being a geometrically scaled-similar sublattice of Λ^1), associated with the quantizers $\{Q^j\}_{j=2}^r$, have generator matrices of the form $\boldsymbol{M}_{\Lambda^j} = diag[1/d_1^j, 1/d_2^j]\boldsymbol{M}_{S\Lambda^j}$, where $d_1^j, d_2^j \in \mathcal{Z}_+$. The division by the integers $\{d_1^j, d_2^j\}$ ensures that the Voronoi cells $\{V_i^Q\}$ will keep a periodic structure, which is still determined by $V_o^{S\Lambda^j}$ (see Fig. 4).

Definition 2 *Given a set of lattices Λ^j $j = 1, \cdots, r$, we define the coincidence site lattice (CSL) Λ^{CSL} as:*

$$\Lambda^{CSL} = \Lambda^1 \cap \Lambda^2 \cap \cdots \cap \Lambda^r \tag{3}$$

and thus, it is the finest common sublattice of all the lattices Λ^j, $j = 1, \cdots, r$.

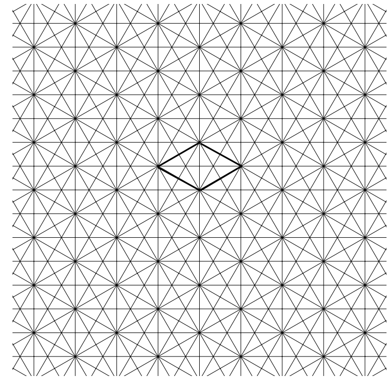

Fig. 5. Example for $r = 3$: Structure of the quantizer Q and unit cell of the structure

The importance of calculating the coincidence site lattice Λ^{CSL} comes from the fact that its fundamental polytope $V_o^{\Lambda^{CSL}}$ is the unit cell that is repeated in the the periodic structure of the resulting quantizer Q. This follows directly from group theory because Λ^{CSL} is the finest common sublattice (subgroup) of all the lattices Λ^j, $j = 1, \cdots, r$.

Example 1 *An example for $r = 3$ is composed by the following tight frame and stepsizes:*

$$\boldsymbol{F} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \cos(\frac{\pi}{6}) & \sin(\frac{\pi}{6}) \\ -\sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \\ \cos(\frac{\pi}{3}) & \sin(\frac{\pi}{3}) \\ -\sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{pmatrix} \tag{4}$$

$$\begin{aligned} &\beta = \frac{1}{\sqrt{3}}, && \Delta_2^1 = \beta\Delta_1^1 = \frac{1}{\sqrt{3}}\Delta_1^1 \\ &\Delta_1^2 = \frac{1}{2}\left(\frac{1}{\cos(\frac{\pi}{6})}\right)\Delta_1^1, && \Delta_2^2 = \frac{1}{2}\left(\frac{3}{\cos(\frac{\pi}{6})}\right)\left(\frac{1}{\sqrt{3}}\right)\Delta_1^1 \\ &\Delta_1^3 = \frac{1}{2}\left(\frac{1}{\cos(\frac{\pi}{3})}\right)\Delta_1^1, && \Delta_2^3 = \frac{1}{2}\left(\frac{1}{\cos(\frac{\pi}{3})}\right)\left(\frac{1}{\sqrt{3}}\right)\Delta_1^1 \end{aligned}$$

Fig. 5 shows the unit cell that is repeated periodically (fundamental polytope of Λ^{CSL}) and the resulting Voronoi cells of the final quantizer Q.

The extension of periodicity for more general redundant families and for higher dimensions is given in [2].

3.2. Consistent reconstruction in Periodic Quantizers

Let $\Phi = \{\{\boldsymbol{\varphi}_i^j\}_{i=1}^N\}_{j=1}^r$ be a tight frame of redundancy r which is composed of r orthogonal bases. It turns out that a necessary condition to have consistency under linear reconstruction for this tight frame is that the quantizer Q has to be periodic. This result is stated in Theorem 1, which is proved in [2].

Theorem 1 *If Q is a non-periodic quantizer in $\mathcal{R}^N$, then it is always possible to find a linearly inconsistent cell, and hence, Q is a quantizer which is not consistent linearly. Therefore, periodicity in a quantizer Q is a necessary condition to achieve consistency under linear reconstruction.*

A sketch of the proof of Theorem 1 is as follows. When there is no periodicity in the partition defined by a quantizer Q, the vertices of any two lattices Λ^{j_1} and Λ^{j_2} can have arbitrary relative positions. When this occurs it is always possible to find a situation where the relative position of the lattices guarantees that an inconsistent linear reconstruction occurs. Notice that since in a periodic quantizer Q there are only a finite number of distinct Voronoi cells $\{V_i^Q\}$, to check if consistency is satisfied linearly, we only need to check on the Voronoi cells contained inside the fundamental polytope of the coincidence site lattice Λ^{CSL}. An example of linear consistency for $r = 2$ is shown in Fig. 3.

Given a periodic quantizer Q, it is also possible to reconstruct efficiently and accurately by using a look-up table of small size. Assume, for simplicity in the discussion, that $N = 2$ and let P_o be the smallest rectangular polytope, which is a basic unit cell for the partition defined by Q (see Fig. 6). The basic idea is that given any Voronoi cell V_i^Q it is possible to find very fast (floor operation and traslation) the equivalent cell $V_{i_o}^Q$ which is inside P_o. Given an input signal $\boldsymbol{x}$, a reconstruction vector $\hat{\boldsymbol{x}}_{\boldsymbol{o}} \in V_{i_o}^Q$ is read from a look-up table and finally this reconstruction vector is translated back into the proper cell V_i^Q. The important advantage provided by the periodicity is that if the periodic quantizer Q is well designed, the size of the look-up table can be made very small, and does not increase with the rate of the quantizer Q.

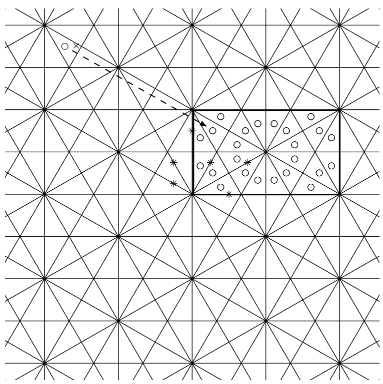

Fig. 6. Reconstruction algorithm based on look-up table: $'\circ'$ represents reconstruction vectors, $'*'$ represents the values of the quantized coefficients which define the equivalent cell in the unit cell P_o, 'x' represents the input vector. All the information is first translated to the unit cell P_o, then the reconstruction vector of the equivalent cell is read, and finally it is translated back to the proper cell

4. EXPERIMENTAL RESULTS

We have compared in $\mathcal{R}^2$ linear reconstruction (with equal stepsizes) and reconstruction based on periodic quantizers (with different stepsizes) using the look-up table scheme, with an input source being a two dimensional Gaussian distribution $\mathcal{N}(\mathbf{0}, \sigma^2 \boldsymbol{I})$ with $\sigma = 0.3$. Although we could have also compared linear reconstruction using both a periodic quantizer and a non-periodic quantizer, we have used a look-up table scheme because for the specific designs that have been used, it is possible to reconstruct linearly with the centroids (as in the look-up table) by using different weights in the linear reconstruction. The comparison has been made by fixing the total rate, which is calculated assuming that all the coefficients are encoded independently with a fixed-length encoding. Actually, this can be seen as being equivalent to making the comparison when the density of points in the space is the same. Let $S = \{\Delta_i^j\}$ be the set of stepsizes used by a periodic quantizer Q and Δ the stepsize used (to quantize all the coefficients of the frame) by another non-periodic quantizer Q'. Notice that each stepsize Δ_i^j of Q can be expressed as $\Delta_i^j = \alpha_i^j \Delta_1^1$, for some $\alpha_i^j \in \mathcal{R}$. The (fixed-length) rate corresponding to each stepsize Δ_i^j can be measured (associated with the density implied by Δ_i^j) as $\log_2(1/\Delta_i^j)$. In order to have the same total rate in both quantizers Q and Q', we need the following condition:

$$\sum_{i,j} \log_2 \left(\frac{1}{\Delta_i^j} \right) = rN \log_2 \left(\frac{1}{\Delta} \right) \Rightarrow \Delta_1^1 = \left(\frac{1}{\prod_{i,j} \alpha_i^j} \right)^{\frac{1}{rN}} \Delta$$

In this way, we can perform a comparison at each value of the stepsize Δ. For each value of Δ, the set S of stepsizes $\{\Delta_i^j\}$ is calculated and the MSE is measured. Fig. 7 represents 2 comparisons,

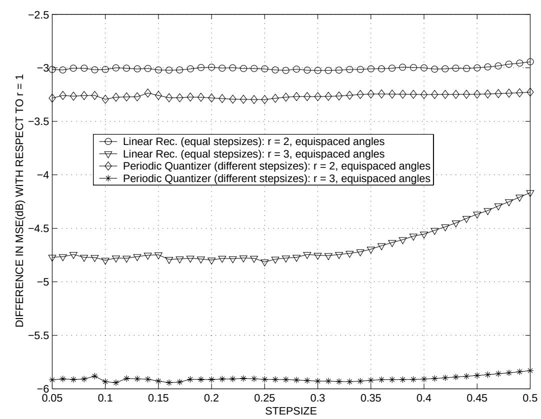

Fig. 7. Comparison between linear reconstruction with equal stepsizes and reconstruction based on look-up table for a periodic quantizer

for $r = 2$ ($\beta = 1$, $\tan(\theta) = 1$ and $\Delta_1^2 = \Delta_2^2 = \sqrt{2}\Delta_1^1$), and $r = 3$ (represented in Fig. 5), where a gain is clearly observed.

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