DESIGN OF 2-BAND ORTHOGONAL NEAR-SYMMETRIC CQF

A. Farras Abdelnour and Ivan W. Selesnick

Polytechnic University
Electrical Engineering
6 Metrotech Center, Brooklyn, NY 11201-3840
Tel: (718) 260-3416, Fax: (718) 260-3906
$\{$*farras*$|$*selesi*$\}$*@taco.poly.edu*

ABSTRACT

The FIR 2-band wavelets have found wide applications in practice. One of their disadvantages, however, is that they cannot be made both symmetric and orthogonal. There have been some works on filters which are orthogonal and nearly symmetric. In this paper, Gröbner methods are used to design orthogonal filters with a subset of exactly symmetric coefficients of various lengths, as opposed to nearly symmetric coefficients.

1. INTRODUCTION

Conjugate quadrature filters (CQF) are filter banks leading to perfect reconstruction. They were first suggested in a paper by Smith and Barnwell [8]. It is a well known fact that for 2-band FIR filters one cannot obtain both orthogonality and symmetry, except for the trivial case of Haar wavelet. Symmetry is desirable in such applications as image processing. One solution has been to drop orthogonality and replace it with biorthogonality in favor of symmetry. But orthogonality is also desired, for example in noise removal applications where noise is not to be further correlated. One alternative would be to make an orthogonal wavelet and accompanying scaling function as symmetric as possible. The idea of 2-band orthogonal filters made almost symmetric has been used in [3, 7, 9, 10]. In the more recent paper by Monzón and Beylkin [6], nearly interpolating properties in Coifman-like filters are sought in addition to approximate symmetry. None of the coefficients of the proposed filters are strictly symmetric. Rather, they are approximately so. In this paper, **Singular** [4] is used to find Gröbner bases [2] of filters with a subset of exactly symmetric coefficients. For K varying from 2 to 5 it was possible to find filters that are very nearly symmetric, with a subset of the coefficients being exactly symmetric while maintaining orthogonality. In some instances the non-symmetric coefficients are very small, possibly of negligible value. The accompanying group delay is found to be nearly flat or varying within a fairly narrow range though in some cases the variation is more pronounced. All filters are orthogonal, near symmetric, with two channels. The resulting filters are compared with generalized Coifman filters found in [9] by D. Wei and A. C. Bovik. It is shown that the Gröbner designed filters offer more flexibility for a given K and more symmetry for a given support. A smoothness coefficient ν_2 discussed in [5] will be used to evaluate the differentiability of the scaling functions corresponding to the various filters found.

2. PRELIMINARIES

Consider a 2-band scaling function $\phi(.)$ defined as:

$$\phi(t) = \sqrt{2}\sum_k h_0(k)\phi(2t-k)$$

where h_0 is a lowpass filter having at least one zero at $z = -1$ and satisfying the condition $\sum_n h_0(n) = \sqrt{2}$. The accompanying wavelet is defined as:

$$\psi(t) = \sqrt{2}\sum_k h_1(k)\phi(2t-k)$$

with $h_1(n)$ the corresponding highpass filter defined as:

$$h_1(n) = (-1)^n h_0(N-1-n)$$

where N is the support of h_0. Then, to generate an orthogonal space spanned by $\phi(.)$ and $\psi(.)$, it is necessary that h_0 be orthogonal to its own shifts by $2k$ $\forall k \in \mathbb{N}$, or

$$\sum_n h_0(n)h_0(n-2k) = \delta(k)$$

It can be shown that the above condition rules out symmetry except for the Haar wavelet [3].

3. NEAR SYMMETRIC FILTERS

The idea is to design lowpass filters with a given number of symmetric coefficients, K zeros at π, and the smallest possible support, N. The resulting filter will of course be only nearly symmetric but, as will be shown, it is possible to design 2-band filters with symmetric subset of coefficients and very small non-symmetric coefficients. The method of Gröbner basis has the advantage of allowing one to explicitly impose the desired properties of the coefficients such as orthogonality and (in our case) partial symmetry. For instance, a typical filter of the partial symmetry sought is of the form $h_0 = [a\ b\ b\ a\ c\ d]$ where $[a\ b\ b\ a]$ is an exactly symmetric part of h_0 and $[c\ d]$ are non-symmetric preferrably of small value coefficients.

It was possible to find filters of both even and odd symmetric lengths within the overall even length of the filter.

3.1. Minimum Support

For a near symmetric lowpass filter with K zeros at $z = -1$, an overall length of N coefficients, and a subset of L symmetric coefficients, we consider the minimum length of a filter satisfying orthogonality as well as near symmetry. Then we need $\frac{N}{2}$ coefficients to satisfy orthogonality condition for the 2-band case and K coefficients for the regularity condition [1]. Additionally, L symmetric coefficients require $L/2$ degrees of freedom for L even, and $(L-1)/2$ for L odd. In either case, this leaves us with $N/2+K$ degrees of freedom for both orthogonality and regularity, or for $L \in 2\mathbb{N}$ we have

$$N - L/2 = N/2 + K$$

and we have

$$N = 2K + L, \quad L \in 2\mathbb{N} \tag{1}$$

Similarly, for $L \in 2\mathbb{N}+1$ we have,

$$N - (L-1)/2 = N/2 + K$$

or

$$N = 2K + L - 1, \quad L \in 2\mathbb{N} + 1$$

Therefore the support of h_0 depends directly on the regularity K and L, the length of the symmetric part. However, one exception to the above equations stands out for the case of $K = 2$, $N = 8$, and $L = 6$ where equation (1) becomes merely an upper bound. Now, if a filter is of overall length N with L symmetric coefficients of even length and regularity K then there exist $K+1$ distinct near symmetric filters of length L for L even. Similarly, we have K distinct filters with an odd number of symmetric coefficients L. It is to be noted that the above results have been confirmed for filter supports up to $\mathrm{supp}(h_0) = 14$.

3.2. Measure of Symmetry

As a measure of symmetry criterion, group delay of h_0 is computed. In the case of symmetry, or $L = N$, the group delay is simply a constant. Various definitions of phase distortion have been used in published papers, see [3, 9, 10]. In this paper we will define group delay error as follows:

$$e = \int_0^{\pi/2} |\tau(\omega) - \tau_0| \, \mathrm{d}\omega \tag{2}$$

where $\tau(\omega)$ is the group delay of the lowpass filter, defined as $\tau(\omega) = -\frac{\mathrm{d}\theta(\omega)}{\mathrm{d}\omega}$, with $\theta(\omega)$ the phase of $H_0(\omega)$, the discrete time Fourier transform of h_0. Also, τ_0 is the average group delay over the interval $[0, \frac{\pi}{2}]$. The integral is evaluated only over the passband interval as the group delay behavior over the stop band is of little relevance. Equation (2) can be approximated as a summation,

$$e \simeq \frac{1}{l} \sum_{n=0}^{l-1} |\tau(\frac{\pi n}{2l}) - \tau_0| \tag{3}$$

where we have l points equally distributed over $[0, \frac{\pi}{2}]$ and τ_0 is the mean value defined as $\frac{1}{l}\sum_n \tau(\frac{\pi n}{2l})$ Obviously, given a number of coefficients, the more symmetric coefficients we have, the closer e is to zero.

3.3. Examples

The use of Gröbner methods made it possible to find filters with symmetric part of different lengths and various centers of symmetry. Several filters have been found with various values of L and K. It was noted that the increase in K requires a proportional increase in L before any symmetric behavior can be observed. As can be seen from table (1), the smoothness coefficient, ν_2, for even L is higher for a given K than the filters with odd L and comparable length N, as shown in table (2). Next, table (3) reflects various degrees of symmetry for different values of K and L. We note that fairly small errors of symmetry have been achieved.

Table 1: ν_2, smoothness coefficient for various values of K and L even, $N = 2K + L$

		K			
		2	3	4	5
L	2	1.1181	1.4970	1.9663	2.3670
	4	1.5094	1.5231	2.0350	2.3030
	6	1.5094	1.5006	2.2745	
	8	1.4649	1.4711		

Table 2: ν_2, smoothness coefficient for K varying from 2 to 4 and L odd, $N = 2K + L - 1$

		K		
		2	3	4
L	3	1.0171	1.6643	1.7882
	5	1.0307	1.7561	2.0212
	7	1.0970	1.8429	
	9	1.1247	2.0807	

Table 3: Error coefficients for various values of K and L.

		K			
		2	3	4	5
L	2	0.116700	0.172300	0.086000	0.096400
	3	0.019800	0.078900	0.213200	
	4	0.019100	0.017200	0.078400	0.057100
	5	0.001600	0.043100	0.029800	
	6	0.019100	0.002700	0.018300	
	7	0.000117	0.014000		
	8	0.001100	0.000840		
	9	0.000005	0.005200		

3.4. Symmetric part of even length

Consider the case of a filter h_0 with $K = 2$ and $L = 6$ and overall support of $\mathrm{supp}(h_0) = 8$. This filter in particular is the only one not to obey equation (1), with $2K + L = 10 > 8$. The corresponding coefficients are as follows:

$$h_0 = \begin{bmatrix} -A \\ A \\ 4A+B \\ 4A+B \\ A \\ -A \\ 4A-B \\ 4A-B \end{bmatrix}$$

with $A = \sqrt{2}/16$, $B = \sqrt{30}/16$. Notice the symmetry in the first six coefficients. The filter smoothness coefficient is given by $\nu_2 = 1.5094$ and symmetry error is given by $e = 0.0191$. We note that the value of ν_2 is influenced by a zero located at $z = -0.9004$, in addition to the two zeros located at $z = -1$. As another example of a filter with even-length symmetric part, consider a filter with $K = 3$, $L = 8$, and overall support $2K + L = 14$. Then we have the error $e = 8.4 \times 10^{-4}$, a highly symmetric filter in this case. The corresponding smoothness coefficient is $\nu_2 = 1.4711$. Figure (1) shows the filter's impulse response as well as the corresponding scaling function.

3.5. Symmetric part of odd length

As an example of a filter with a subset of coefficients with odd-length symmetry, we consider a filter with $K = 2$, $L = 9$, and overall length $2K + L - 1 = 12$. Then, the symmetry error is $e = 5.3910 \times 10^{-6}$. The corresponding smoothness coefficient is then $\nu_2 = 1.0374$. See figure (3) for the impulse response and scaling function.

3.6. comparison with published results

The filters designed in [9] are compared with filters found using Gröbner methods. As an example, consider the case of a filter with even near-symmetry and $\text{supp}(h_0) = 12$, the Gröbner offers a symmetry-improved design over the one published in [9]. The error coefficient in Wai and Bovik filters was found to be $e = 0.1142$ and $\nu_2 = 1.8210$. Compare with Gröbner designed filter of same support but with four symmetric coefficients and error $e = 0.0784$ and $\nu_2 = 2.0351$, an improvement in both parameters. See figure (2) for the resulting filters. Now consider $\text{supp}(h_0) = 12$ with odd number of symmetric coefficients. In this case the Wei and Bovik filter the error of symmetry coefficient is $e = 0.0259$ and degree of smoothness $\nu_2 = 1.8327$. Compare with the case of a Gröbner designed filter of the same support and seven symmetric coefficients. In this case we have $e = 0.0140$ and $\nu_2 = 1.8429$, an improvement in symmetry and slight improvement in smoothness. See figure (4).

4. CONCLUSION

The degrees of freedom in the filters designed in [9] are used to satisfy orthogonality, near symmetry, and the coiflet condition of vanishing moments on both the scaling function and the wavelet. Using Gröbner methods it was possible to design orthogonal filters with nearly symmetric properties and a subset of exactly symmetric filters. Those filters offer improved symmetry for a given support and in some instances improved smoothness as well.

Table 4: Coefficients of various filters

$K=3, L=8$	$K=2, L=9$	$K=4, L=4$	$K=3, L=7$
0.000522	0.000042	-0.005941	0.015864
0.004477	0.000776	0.026294	-0.050704
0.006199	-0.009253	0.034885	-0.072207
-0.086052	-0.073222	-0.085213	0.401755
0.085824	0.362766	0.111204	0.812841
0.696542	0.852001	0.688258	0.401755
0.696542	0.362766	0.688258	-0.072207
0.085824	-0.073222	0.111204	-0.050704
-0.086052	-0.009253	-0.131445	0.024837
0.006199	0.000776	-0.035729	0.005638
0.004232	-0.000039	0.010145	-0.002021
0.000096	-0.000002	0.002292	-0.000632
-0.000162	0	0	0
0.000019	0	0	0

5. REFERENCES

[1] C. S. Burrus, R. A. Gopinath, and H. Guo. *Introduction to Wavelets and Wavelet Transforms*. Prentice Hall, 1997.

[2] D. Cox, J. Little, and D. O'Shea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. Springer-Verlag, 1991.

[3] I. Daubechies. *Ten Lectures On Wavelets*. SIAM, 1992.

[4] G.-M. Greuel, G. Pfister, and H. Schönemann. Singular Reference Manual. In *Reports On Computer Algebra*, number 12. Centre for Computer Algebra, University of Kaiserslautern, May 1997. *http://www.singular.uni-kl.de/*.

[5] B. Han. Symmetric orthonormal scaling functions and wavelets with dilation factor 4. *Advances in Computational Mathematics*, (8):221–247, 1998.

[6] L. Monzón and G. Beylkin. Compactly supported wavelets based on almost interpolating and nearly linear phase filters (coiflets). *Applied and Computational Harmonic Analysis*, 7:184–210, 1999.

[7] I. Selesnick, J. E. Odegard, and C. S. Burrus. Nearly symmetric orthogonal wavelets with non-integer dc group-delay. In *Digital Signal Processing Workshop Proceedings*, pages 431–434, September 1-4 1996.

[8] M. J. T. Smith and T. P. Barnwell III. Exact reconstruction for tree-structured subband coders. *IEEE Trans. on Acoust., Speech, Signal Proc.*, 34(3):431–441, June 1986.

[9] D. Wei and A. C. Bovik. Generalized coiflets with nonzero-centered vanishing moments. *IEEE Trans. on Circuits and Systems II*, 45:988–1001, August 1998.

[10] Y. Zhao and M.N. S. Swamy. Design of least asymmetric compactly supported orthogonal wavelets via optimization. In *Electrical and Computer Engineering, 1999 IEEE Canadian Conference on*, volume 2, pages 817 – 820, May 1999.

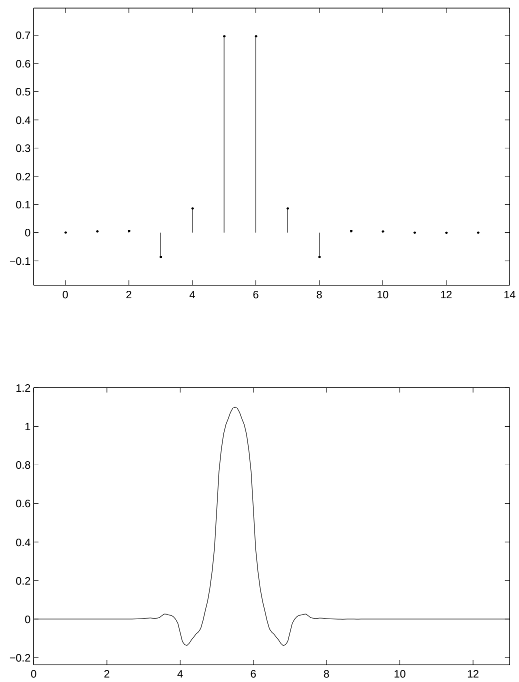

Figure 1: $L = 8$ and $K = 3$. Filter impulse response and resulting scaling function.

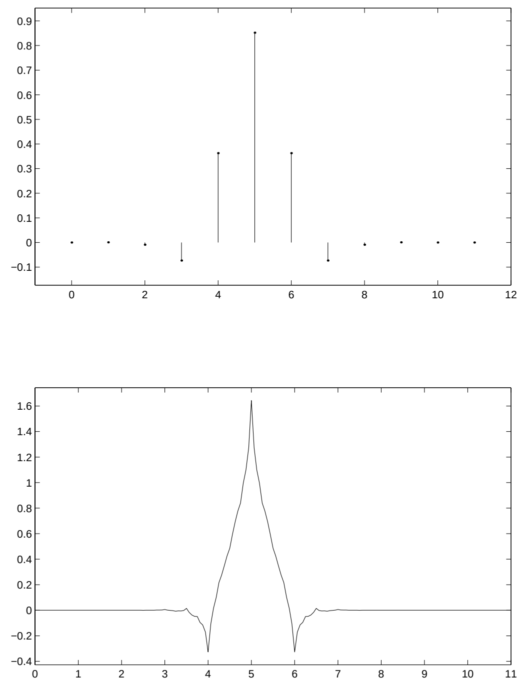

Figure 3: $L = 9$ and $K = 2$. Filter impulse response and resulting scaling function.

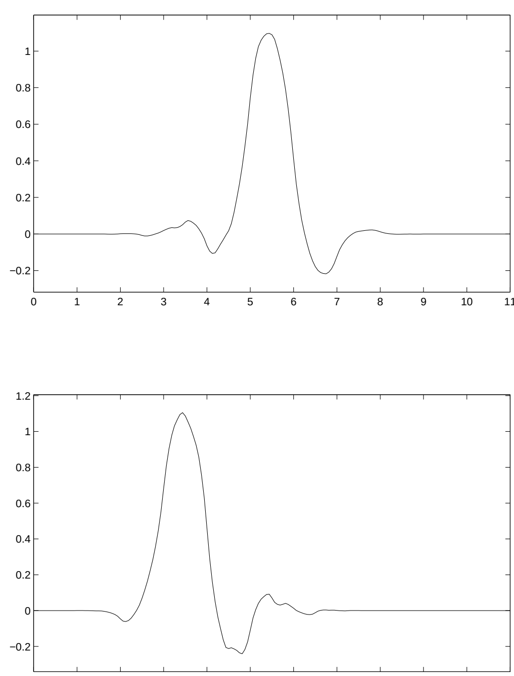

Figure 2: $K = 4$, scaling functions of support 12. Gröbner design, top; Wei and Bovik design, above.

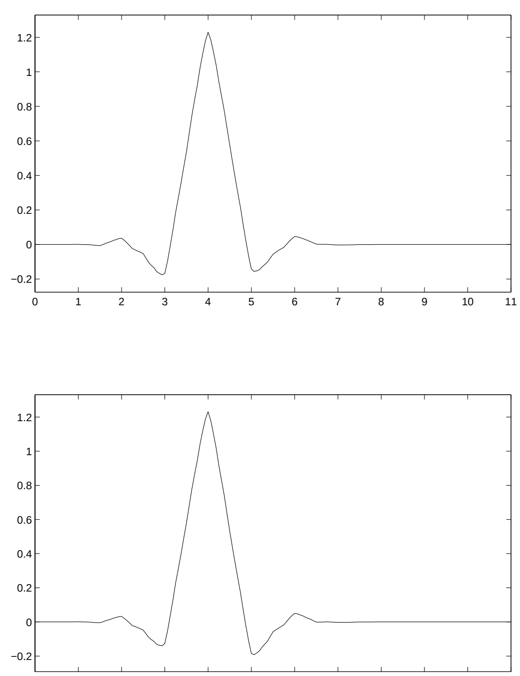

Figure 4: $K = 4$, scaling functions of support 12. Gröbner design, top; Wei and Bovik design, above.