PEAK LOCATIONS IN ALL-PASS SIGNALS
THE MAKHOUL CONJECTURE CHALLENGE

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ABSTRACT

In [1] the Makhoul Conjecture Challenge was published. To answer was the question whether or not the location of the peak of a digital stable all-pass filter lies in [0, 2p-1], where p is the order of the all-pass filter. In this paper we construct numerous counter-examples, prove a new theorem stating that there is at least an upper bound on the order of $p^{3/2}$ for the location of the peak, and discuss the algebraic structure of all-pass filters and their impulse responses. The paper is heavily based on an experimental approach.

1. INTRODUCTION

Let X(z) be a digital all-pass filter of order p with real coefficients.

(1) $$X(z) = \frac{a_p + a_{p-1}z^{-1} + ... + z^{-p}}{1 + a_1 z^{-1} + ... + a_p z^{-p}} = \frac{z^{-p}A(z^{-1})}{A(z)}$$

The zeros of the polynomial A(z) are strictly inside the unit circle, i.e. X(z) is a stable digital filter. Let m be a value for the index n for which the impulse response x[n] of X(z) has its maximum amplitude. In [1, 2, 3] Makhoul conjectured that $0 \le m \le 2p-1$.

(2) For $a_1 = a_2 = \ldots = a_p = 0$ we have $X(z) = z^{-p}$

which results in m=p. This equation can be generalized. To understand this, it can be shown that

(3) $$p = \sum_{n=0}^{\infty} n h[n]^2$$

for any all-pass filter (1), where h[n] is the filter's impulse response. Furthermore,

(4) $$1 = \sum_{n=0}^{\infty} h[n]^2$$

Based on (3) and (4) Makhoul and Steinhardt [2] proved that

(A) The minimal absolute value h_p of the peak of a stable all-pass filter of order p satisfies the inequality $h_p \le 1/\sqrt{2p+1}$.

(B) The maximum possible location of the signal peak is on the order of p(p+3)/2.

The proofs deal with the more general case of causal signals with given average delay but the results are in particular valid for the impulse responses of stable all-pass filters. It was very unlikely that impulse responses of stable all-pass filters can even come close to these bounds (A) and (B) because these numbers are valid for a much broader class of signals. This gave rise to the mentioned conjecture that the maximum possible location of the impulse response peak of a stable all-pass filter lies in [0, 2p-1] where p is the order. The IEEE Signal Processing Magazine published this "Makhoul Conjecture Challenge" in the May 2000 issue. Either a proof or a counter-example was regarded as a complete solution. The conjecture itself was also based on numerical experiments, where orders less than 6 were investigated extensively. It was highly unlikely that a counter-example could be found in this region.

Our approach is as follows. We start with some known results regarding the phase behavior of stable all-pass filters (1) and derive a new upper bound for the magnitude of impulse response signals. Based on this, we show that (B) can be replaced with a better upper bound, namely:

(B') The maximum possible location of the impulse response peak of a stable all-pass filter is on the order of $p\sqrt{2p+1}$.

We present some observations resulting from computer experiments. Among them we demonstrate that Makhoul's conjecture for p>5 is false. We also present construction schemes for stable all-pass filters that produce peak locations far beyond the original 2p bound. All experiments are based on the graphical language LabVIEW.

There are also interesting algebraic and group theoretical properties of all-pass filters. These could, at least in principle, be used to construct new all-pass filters based on given structures. We finish the paper with open questions and suggestions. The field of all-pass filters offers many challenges for both computational and theoretical approaches.

2. PROPERTIES OF ALL-PASS SYSTEMS

A stable system function of the form (see for this and the following formulas [4])

(5) $$H_1(z) = \frac{-a^* + z^{-1}}{1 - az^{-1}}$$

where *a* is a complex number with magnitude less than 1 and a^* the complex conjugate of *a*, has a frequency response magnitude that is independent of the underlying frequency. **Figure 1** depicts the phase response, distribution of zeros, and group delay of a filter composed of 15 such systems. Depending on the a-value, the peaks of the phase response are more or less pronounced. The phase response of (5) is

(6) $$\arg(H_1(z)_{z=e^{j\omega}}) = -\omega - 2\arctan(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)})$$

where a=re$^{j\omega}$. The phase response of a second order real all-pass system with poles at z=re$^{j\theta}$ and z=re$^{-j\theta}$ is

(7)
$$\arg(H_2(z)_{z=e^{j\omega}}) = -2\omega - 2\arctan(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}) - 2\arctan(\frac{r\sin(\omega+\theta)}{1-r\cos(\omega+\theta)})$$

The group delay of a first order complex all-pass filter with pole at z=re$^{j\theta}$ is:

(8) $$grd(H_1(z)_{z=e^{j\omega}}) = \frac{1-r^2}{1+r^2-2r\cos(\omega-\theta)} = \frac{1-r^2}{\left|1-re^{i\theta}e^{-j\omega}\right|}$$

The group delay of a pth order stable all-pass filter is simply the sum of components presented in (8). In particular, (8) is a positive function, i.e. the phase response is always negative and monotone decreasing.

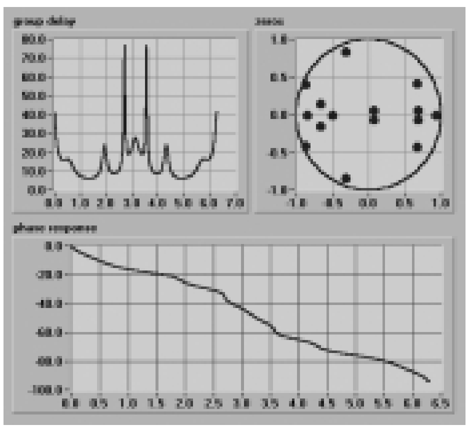

Figure 1: Group delay , phase response, and distribution of zeros of a typical real all-pass filter.

3. A NEW UPPER BOUND FOR THE MAXIMUM LOCATION OF COMPLEX ALL-PASS FILTERS

In this section we develop a new upper bound for the maximum location of complex all-pass filters.

Lemma 1: Let f: [0, 2π] -> [0, 2π] be a continuous real function with f(0)=0 and f(2π)=2π. Let f furthermore be monotone. The complex coefficients

$$h[n] = \frac{1}{2\pi}\int_0^{2\pi} dt \exp(jf(t))\exp(jnt)$$

satisfy the inequalities

$\left|h[n]\right| \le \frac{1}{n}$ for n = 1, 2, …

Proof: All proofs are omitted due to space constraints. Please contact authors for proofs. A. El-Jaroudi [5] provided us with a simplified version of our original proof.

Lemma 1 can be generalized. The proof must be changed only slightly.

Lemma 2: Let f: [0, 2π] -> [0, 2πp] be a continuous real function with f(0)=0 and f(2πp)=2πp where p is a given natural number. Let f furthermore be monotone. The complex coefficients

$$h[n] = \frac{1}{2\pi}\int_0^{2\pi} dt \exp(jf(t))\exp(jnt)$$

satisfy the inequalities $\left|h[n]\right| \le \frac{p}{n}$ for n = 1, 2, …

3.1 Complex All-pass Filters

The group delay of a first order complex all-pass filter with a zero at re$^{j\tau}$ can be computed by $gd(\omega) = (1-r^2)/\left|1-re^{j\tau}e^{-j\omega}\right|$. It follows that the group delay of a generic complex all-pass filter of order p is continuous and strictly positive. The phase function f(ω) is a continuous and monotone decreasing function with the additional properties f(0) = 0 and f(2πp) = 2πp.

Let h[n] be the impulse response of a complex all-pass filter of order p. Then:

(9) $$h[n] = \frac{1}{2\pi}\int_0^{2\pi} dt \exp(jf(t))\exp(jnt)\cdot$$

According to **Lemma 2** it follows that the magnitude of h[n] is bounded by *p/n* for all natural numbers n.

The maximal magnitude of the peak of a signal with average delay p is bounded by the lower limit $1/\sqrt{2p+1}$ (John Makhoul, Allan O. Steinhardt, 1991, [2]). Let n_{max} denote this position. We have

$$\frac{1}{\sqrt{2p+1}} \le \left|h[n_{\max}]\right| \le \frac{p}{n_{\max}} \text{, i.e. } n_{\max} \le p\sqrt{2p+1}\cdot$$

Theorem 1: The impulse response h[n] of a complex all-pass filter has its maximum location in the interval $[0, p\sqrt{2p+1}]$.

4. THE MAKHOUL CONJECTURE

Based on modern programming systems, such as LabVIEW [6], a systematic investigation of the behavior of real and complex all-pass filters is possible. It can be verified that Makhoul's conjecture regarding the maximum location of real all-pass filters seems to be true for p = 1, 2, 3, 4, and 5. We don't know whether or not a formal proof of the conjecture for the numbers p = 3, 4, and 5 is known. The first counter-example we found during systematic investigations of all stable real all-pass filters had an order p = 8 and produced a peak at the position 16. This value is beyond the conjectured upper bound of 2p-1. Further test runs showed that even in the case of p = 6 a counter-example can be generated.

Table 1 gives an overview of observed locations of peaks of impulse responses. It is not known whether these values represent the maximum peak position of a given order (except p=1,2). In all cases, test runs on the order of some millions each were performed.

order p	location	order p	location	order p	location
1	1	8	17	15	33
2	3	9	19	16	36
3	5	10	21	17	39
4	7	11	23	18	43
5	9	12	26	19	45
6	12	13	28		
7	14	14	30		

Table 1: Observed locations of peaks of real and stable all-pass filters.

As **Figure 2** demonstrates, the distributions of poles of at least near optimal real and stable all-pass filters seem to prefer certain patterns. By optimality we imply a filter that tries to maximize the peak location. The next section further elaborates this observation.

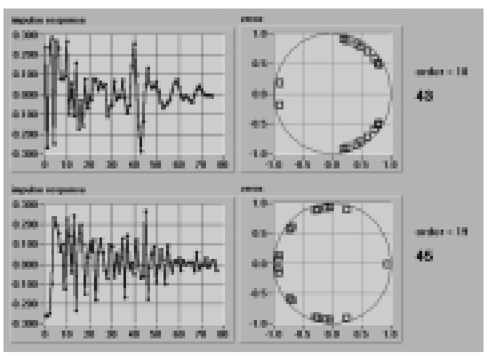

Figure 2: Observed peaks for *p = 18* and *p = 19*. The distributions of the underlying poles suggest the investigation of certain patterns. See section 5 for more details.

5. MORE SOPHISTICATED COUNTER-EXAMPLES

The structure of the group delay function according to formula (8) in combination with (9) and numerical experiments suggest that more pronounced peak delays can be expected when the filter-poles are:

(I) Always in the immediate neighborhood of the border of the unit circle (but not too close)
(II) Distributed in a chirp-like pattern
(III) Approaching the border of the unit circle, i.e. at least some poles are extremely close to the border

The second condition can be understood based on formula (9). This expression can be regarded as the Fourier transform of certain functions with a very specific structure. The goal for successfully generating a quasi-optimal filter is to avoid a rapidly vanishing set of Fourier coefficients. On the other hand, according to (9) the functions under consideration are smooth. All these arguments suggest the use of chirp-like distributions, implying a clustering scheme of the poles according to **Figure 3**.

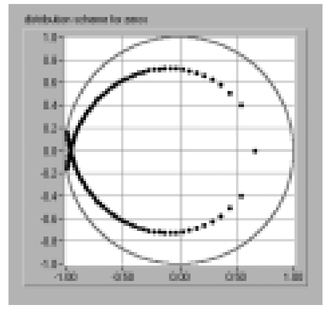

Figure 3: Distribution scheme according to (I)-(III).

Experimental results ([7]) confirm this idea. **Figure 4** gives an example of a stable and real-valued all-pass filter of order p = 4000, where the location of the peak is far beyond *2p*. For this specific example it is even beyond *5p*. In this case, the impulse response is clearly divided into a relatively early slowly decaying answer and a very late second peak followed by some smaller peaks. The signal decays rapidly after passing the peak. **Figure 4** also magnifies the critical region beyond a location of 20,000 and depicts the distribution of poles near the point (-1,0). The solid line represents the unit circle and the dotted lines stand for poles.

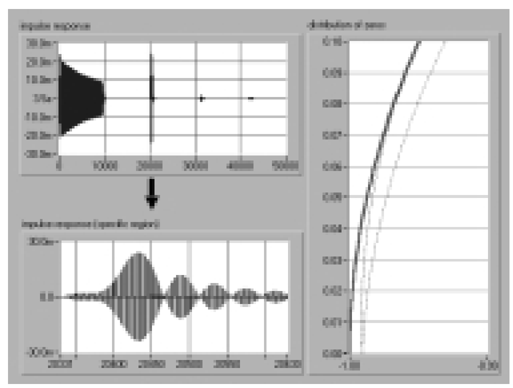

Figure 4: This example breaks the *5p* limit ([7]). It is based on the scheme (I)-(III). The parameters controlling (I)-(III) must be chosen with care.

We don't know whether better strategies can be found. The same is true for the more general case of complex-valued stable all-pass filters.

The computation of the power spectrum of the impulse response according to **Figure 4** reveals that the different active zones (i.e. absolute response values beyond a small number) represent different spectral regions. This might be a clue on how to construct even more impressive peak delays.

The behavior of these counter-examples is interesting. A stable real all-pass filter of order *p* delays exp-functions no more than *2p* time steps. To a certain extent, an impulse response, though a linear combination of these exp-functions, behaves very differently. The nonlinear character of the phase response leads to distortions and can help in moving the peak farther away.

6. ADDITIONAL ALGEBRAIC AND GROUP THEORETICAL PROPERTIES

We discuss further results that could be useful to gain a better understanding of extreme locations of real or complex stable all-pass filters. (For proofs please contact the authors).

Theorem 2: Let *h[0]*, *h[1]*, *h[2]*, … be the impulse response of a real stable all-pass filter of order *p*. Then the following equations hold true for all natural numbers *k = 0, 1, 2, …*

$$\sum_{n=0}^{\infty} h[n]h[n+k] = \delta_k \tag{10}$$

$$\sum_{n=0}^{\infty} nh[n]h[n] = p \tag{11}$$

where $\delta_k = 1$ if and only if k = 0 and $\delta_k = 0$ in all the other cases.

A similar theorem can be formulated for complex stable all-pass filters. Makhoul's and Steinhardt's results in [2] are based on **Theorem 2** where only k=0 was used, cf. also (3) and (4). It might be an advantage to consider the use of **Theorem 2** for all k. See also (c) and (h) in section 7.

A systematic search for extreme peak location behavior of stable all-pass filters could be based on appropriate combinations of given all-pass filters. To demonstrate this idea, let

$G_a(z) = \dfrac{-a^* + z}{1 - az}$ and $G_b(z) = \dfrac{-b^* + z}{1 - bz}$

be the algebraic representation of two all-pass filters of order 1 each according to (5) where the magnitudes of a and b are less than 1 and z^{-1} is replaced with z. It turns out that the concatenation $G_a(G_b(z))$ has a specific form, namely,

$$G_a(G_b(z)) = \frac{-a^* + \dfrac{-b^* + z}{1 - bz}}{1 - a\dfrac{-b^* + z}{1 - bz}} = \left(\frac{1 + a^* b}{1 + ab^*}\right)\frac{-\left(\dfrac{a^* + b^*}{1 + a^* b}\right) + z}{1 - \left(\dfrac{a + b}{1 + ab^*}\right)z}$$

Ignoring the leading phase shift of $\left(\dfrac{1 + a^* b}{1 + ab^*}\right)$ the group theoretical formula $a \otimes b = \dfrac{a + b}{1 + ab^*}$ can be derived, i.e. $G_a(G_b) = G_{a \otimes b}$. For all-pass filter orders beyond 1 the situation is more complicated and the pure group theoretical background cannot be maintained. It can be shown that the following theorem holds true.

Theorem 3: Let be given two stable complex all-pass filters $H_1(z)$ and $H_2(z)$ of order *p* and *q*, respectively. Then $H_1(H_2(z))$ is a stable complex all-pass filter of order *pq*.

There is another more obvious algebraic operation that produces stable all-pass filters as long as the operands are stable all-pass filters.

Theorem 4: Let be given two stable complex all-pass filters $H_1(z)$ and $H_2(z)$ of order *p* and *q*, respectively. Then $H_1(z)H_2(z)$ is a stable complex all-pass filter of order *p+q*.

It remains to be seen whether the resulting algebraic structures can be used to construct specific all-pass filters with large peak delays based on given all-pass filters with the same behavior.

7. REMARKS AND OPEN QUESTIONS

We simply list here some remarks and open questions that could be of interest for further investigations.

a) Which of the numbers in **Table 1** represent the real upper bounds?
b) Is the situation different when complex filters are considered? Surprisingly, the new degrees of freedom offered by complex coefficients in (1) don't seem to produce much better maximum location values.
c) The results in [2] are based on some specific algebraic properties mentioned in section 6. Can one find sharper bounds when the whole set of algebraic relationships according to **Theorem 2** is used?
d) Are there real stable all-pass filters generating maximum peak locations beyond c*p where c is an arbitrary but fixed natural number? According to the results in section 5 one can at least choose c = 5. Furthermore, is an expression on the order of log(p)*p a (sharp) upper bound for this location?
e) Can one use the spectral decomposition mentioned in section 5 to generate even more impressive examples?
f) Are there potential applications for stable implementations of these extreme real all-pass filters?
g) Relating to (f), is it possible to use such filters for information storage purposes?
h) Do the algebraic formulas (10) and (11) characterize the impulse responses of stable all-pass filters completely?
i) Is Makhoul's conjecture true if the peak value of the all-pass filter is also being minimized ([2])?

Again, the field of all-pass filters offers many challenges for both computational and theoretical approaches.

ACKNOWLEDGEMENT

The authors wish to thank A. Moriat, J. Gerved, A. El-Jaroudi, and J. Makhoul for valuable discussions and assistance during the preparation of the paper.

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