ALLPASS FILTER DESIGN USING PROJECTION-BASED METHOD UNDER GROUP DELAY CONSTRAINTS

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ABSTRACT

A new technique for designing digital allpass IIR filters is proposed. The approach is based on the vector space projection method. Constraint sets, and their associated projectors, that capture the properties of the desired group delay are given. Examples that demonstrate the advantages and flexibility of this method as well as comparisons with a well- known method are furnished

1. INTRODUCTION

Some of the numerous allpass filter applications [1] are: approximation of a prescribed phase, e.g., a linear phase (fractional delay elements); equalization of a phase or group delay of a given system; design of a Hilbert transformer; and design of recursive filters with the desired magnitude response using a parallel allpass structure. A number of authors have approached the allpass filter design problem in the least square sense [2]. The eigenfilter approach for least-squares allpass filter design introduced in [3] is based in formulating the objective function in a quadratic form and obtaining the desired filter as an eigenvector of a suitably defined real, symmetric, positive-definite matrix. In this paper, we will consider a new approach to the design of allpass filters for the group-delay equalization problem and we shall compare the results with the ones of the eigenfilter method [3].

2. VECTOR SPACE PROJECTION METHOD

The *vector space projection method* (VSPM) deals with the problem of finding a mathematical object (for example, a signal, function, image brightness, etc.) that satisfies multiple constraints in a vector space such as the L^2 space od square-integrable functions, the l^2 space of square-summable sequences, or the Euclidean space R^n. The theory was initially developed for *intersecting*, *convex*, constraints sets by Bregman [4] and Gubin et al. [5] and was first applied to image processing by Youla and Webb [6]. In recent years the theory has been extended to non-convex, non-intersecting sets. In particular, Combettes [7], building on the work of Pierra [8], and Levi [9] described an algorithm based on the following theorem:

Theorem. For every $\mathbf{x}_0 \in \mathbf{H}$ and every choice of positive constant $w_1, w_2, \cdots, w_m$ such that $\sum_{i=1}^{m} w_i = 1$, the sequence $\{\mathbf{x}_n\}$ generated by

$$\mathbf{x}_{n+1} = \sum_{i=1}^{m} w_i P_i \mathbf{x}_n, \tag{1}$$

converges weakly to a point $\mathbf{x}^*$ such that $\Phi(\mathbf{x}^*) \equiv \sum_{i=1}^{m} w_i d^2(\mathbf{x}^*, C_i)$ is minimized.

This result applies whether the constraint sets intersect or not. Moreover, it applies whether the sets are convex or not. In this Letter, we exploit the above Theorem to design a class of IIR filters where the appropriate constraints are not necessarily convex, and where the intersection of large number of constraint sets may be the empty set.

For the reader not familiar with the theory of VSPM, an introduction with examples is furnished in [10].

3. ALLPASS FILTER GROUP DELAY

The most general form for the system function of an allpass system with real coefficients, is given by

$$H(z) = \frac{\sum_{i=0}^{N} a_{N-i} z^{-i}}{\sum_{i=0}^{N} a_i z^{-i}}. \tag{2}$$

The phase versus frequency characteristics can be expressed as [10]

$$\theta_A(\omega) = -N\omega + 2\arctan\left\{\frac{\sum_{k=0}^{N} a_k \sin(k\omega)}{\sum_{k=0}^{N} a_k \cos(k\omega)}\right\} \tag{3}$$

$$= -N\omega + 2\arctan\left\{\frac{\mathbf{a}^T\mathbf{s}(\omega)}{\mathbf{a}^T\mathbf{c}(\omega)}\right\},$$

where **a** is the allpass coefficient vector $\mathbf{a} = (a_0, a_1, \cdots, a_N)^T$ and the vectors $\mathbf{s}(\omega)$ and $\mathbf{c}(\omega)$ are given by

$$\begin{aligned}\mathbf{s}(\omega) &= [0 \quad \sin(\omega) \quad \cdots \quad \sin(N\omega)]^T \\ \mathbf{c}(\omega) &= [1 \quad \cos(\omega) \quad \cdots \quad \cos(N\omega)]^T.\end{aligned} \tag{4}$$

After differentiability Eq.(3) with respect to ω we obtain the group delay of the allpass filter

$$\tau_A(\omega) = -\frac{d\theta_A(\omega)}{d\omega} = N - 2\frac{\mathbf{a}^T\mathbf{G}\boldsymbol{\Lambda}\mathbf{a}}{\mathbf{a}^T\mathbf{G}\mathbf{a}}, \tag{5}$$

where $\mathbf{G} \equiv \mathbf{c}(\omega)\mathbf{c}(\omega)^T + \mathbf{s}(\omega)\mathbf{s}(\omega)^T$ and $\boldsymbol{\Lambda} \equiv \mathrm{diag}[0 \quad 1 \quad \cdots \quad N]$.

The phase and group delay of an allpass filter are related to the coefficients in a very nonlinear manner, as the above equations show. This means that one cannot expect as simple a design procedure for computing the appropriate allpass filter coefficient vector **a** as in the case for FIR filters. Instead, one usually uses iterative optimization techniques for minimization of traditional error criteria.

4. DESIGN OF ALLPASS FILTER USING VSPM

Suppose we want to equalize a phase $\theta_P(\omega)$ over the range $0 \le \omega \le \omega_p$. After phase equalization by the allpass filter $\theta_A(\omega)$, the total phase $\theta_T(\omega)$, should be a linear function of frequency i.e.,

$$\theta_T(\omega) \equiv \theta_P(\omega) + \theta_A(\omega) = -K\omega, \tag{6}$$

where $K > 0$ for causality. A question that arises is what should K be? An appropriate value of K must take into consideration questions of tolerable delay, stability, and quality of equalization. The overall group delay τ_T is the negative of the derivative, with respect to frequency, of the total phase, hence

$$\tau_T = -\frac{d\theta_P}{d\omega} - \frac{d\theta_A}{d\omega} = K > 0 \tag{7}$$

However, it is more realistic to introduce a tolerance parameter δ and replace the strict equality of Eq.(7) with

$$K - \delta \le -\frac{d\theta_P}{d\omega} - \frac{d\theta_A}{d\omega} \le K + \delta \tag{8}$$

Causality requires $-d\theta_A/d\omega > 0$. Hence from the RHS of Eq.(8):

$$K \ge \max_{0\le\omega\le\omega_P}\left\{-\frac{d\theta_P}{d\omega} + \left|\frac{d\theta_A}{d\omega}\right| - \delta\right\} \tag{9}$$

so that K is absolutely bounded from below by

$$K_{LB} = \max_{0\le\omega\le\omega_P}\left\{-\frac{d\theta_P}{d\omega} - \delta\right\}. \tag{10}$$

Unfortunately the LHS of Eq.(8) cannot be used to establish an upper bound on K that is not overly pessimistic and may lead to nonsense results.

From Eq.(5) and Eq.(8) we obtain the important result that

$$\underbrace{\frac{-K - \delta - \theta'(\omega) + N}{2}}_{\delta_1(\omega)} < \frac{\mathbf{a}^T\mathbf{G}\boldsymbol{\Lambda}\mathbf{a}}{\mathbf{a}^T\mathbf{G}\mathbf{a}} < \underbrace{\frac{-K + \delta - \theta'(\omega) + N}{2}}_{\delta_2(\omega)}. \tag{11}$$

In solving a problem by VSPM, the key is to define the appropriate constraint sets. In this problem an appropriate cluster of constraints sets are defined by

$$C_i \equiv \left\{\mathbf{a} : \delta_1(\omega_i) \le \frac{\mathbf{a}^T\mathbf{G}_i\boldsymbol{\Lambda}\mathbf{a}}{\mathbf{a}^T\mathbf{G}_i\mathbf{a}} \le \delta_2(\omega_i)\right\}, \tag{12}$$

where $\omega_i = \frac{\omega_p}{M-1} i$, is discretized version of the continuous ω and $i = 0, \cdots, M-1$. In general the sets C_i $i = 0, \cdots, M-1$ are non-convex.

We note that $\mathbf{a}^T\mathbf{G}_i\mathbf{a} \ge 0$. Then, assuming $\mathbf{a}^T\mathbf{G}_i\mathbf{a} \ne 0$ we can rewrite C_i in Eq.(12) as

$$C_i = \{\mathbf{a}\!: \delta_1(\omega).\mathbf{a}^T\mathbf{G}_i\mathbf{a} \le \mathbf{a}^T\mathbf{G}_i\boldsymbol{\Lambda}\mathbf{a} \le \delta_2(\omega).\mathbf{a}^T\mathbf{G}_i\mathbf{a}\} \tag{13}$$

which can be further combined as

$$C_i = \{\mathbf{a}\!:\ \mathbf{a}^T\mathbf{P}_{1i}\mathbf{a} \ge 0 \text{ and } \mathbf{a}^T\mathbf{P}_{2i}\mathbf{a} \le 0\}, \tag{14}$$

where

$$\begin{aligned}\mathbf{P}_{1i} &\equiv \mathbf{G}_i\boldsymbol{\Lambda} - \delta_1(\omega_i)\mathbf{G}_i \\ \mathbf{P}_{2i} &\equiv \mathbf{G}_i\boldsymbol{\Lambda} - \delta_2(\omega_i)\mathbf{G}_i.\end{aligned} \tag{15}$$

Because $\mathbf{G}_i$ is of rank two and $rank(\mathbf{G}_i\boldsymbol{\Lambda}) \le \min(rank\{\mathbf{G}_i\}, rank\{\boldsymbol{\Lambda}\})$ it follows that the asymmetrical matrices $\mathbf{P}_{1i}$ and $\mathbf{P}_{2i}$ are at most of rank two. Indeed, it can be shown, they have rank two. Since real symmetric matrices have the desirable properties of having real eigenvalues and real, orthogonal eigenvectors, we create such matrices from $\mathbf{P}_{1i}$ and $\mathbf{P}_{2i}$ as

$$\mathbf{F}_{1i} = \frac{(\mathbf{P}_{1i} + \mathbf{P}_{1i}^T)}{2} \quad \text{and} \quad \mathbf{F}_{2i} = \frac{(\mathbf{P}_{2i} + \mathbf{P}_{2i}^T)}{2}. \tag{16}$$

$\mathbf{F}_{1i}$ and $\mathbf{F}_{2i}$ are symmetrical and of rank four (in this particular case). The quadratic form generated by $\mathbf{F}_{1i}$ (or $\mathbf{F}_{2i}$) is identical with that generated by $\mathbf{P}_{1i}$ (or $\mathbf{P}_{2i}$). To show this let **F** denote either $\mathbf{F}_{1i}$ or $\mathbf{F}_{2i}$ and let **P** denotes the corresponding $\mathbf{P}_{1i}$ or $\mathbf{P}_{2i}$. Then

$$\mathbf{x}^T\mathbf{F}\mathbf{x}=\mathbf{x}^T\left(\frac{\mathbf{P}+\mathbf{P}^T}{2}\right)\mathbf{x}=\frac{\mathbf{x}^T\mathbf{P}\mathbf{x}}{2}+\frac{(\mathbf{P}\mathbf{x})^T\mathbf{x}}{2}$$
$$=\frac{\mathbf{x}^T\mathbf{P}\mathbf{x}}{2}+\frac{\mathbf{x}^T(\mathbf{P}\mathbf{x})}{2}=\mathbf{x}^T\mathbf{P}\mathbf{x}. \quad (17)$$

With the result in Eq.(17), we can rewrite C_i in Eq.(14) as

$$C_i=\{\mathbf{a}\colon \mathbf{a}^T\mathbf{F}_{1i}\mathbf{a}\geq 0 \text{ and } \mathbf{a}^T\mathbf{F}_{2i}\mathbf{a}\leq 0\}$$
$$=\underbrace{\{\mathbf{a}\colon \mathbf{a}^T\mathbf{F}_{1i}\mathbf{a}\geq 0\}}_{C_{1i}}\cap\underbrace{\{\mathbf{a}\colon \mathbf{a}^T\mathbf{F}_{2i}\mathbf{a}\leq 0\}}_{C_{2i}}. \quad (18)$$

The reason for resolving C_i into an intersection of two simpler sets is that it is easier to compute the projections onto C_{1i} and C_{2i} separately than computing a single projection onto C_i. Once again letting $\mathbf{F}_i$ denote either $\mathbf{F}_{1i}$ or $\mathbf{F}_{2i}$, we note that projecting onto C_{1i} or C_{2i} amounts to finding a point $\mathbf{a}$ on the surface $\mathbf{a}^T\mathbf{F}_i\mathbf{a}=0$ that is *nearest* to an arbitrary point $\mathbf{g}$ outside the set. Since the projectors onto C_{1i} or C_{2i} have identical form, we consider projecting onto the generic set $C_i=\{\mathbf{a}\colon \mathbf{a}^T\mathbf{F}_i\mathbf{a}\geq 0\}$.

Projecting onto $C^{(i)}$. Finding the projection onto $C^{(i)}$ of an arbitrary vector $\mathbf{g}$ involves finding the extremum of the functional $J=\|\mathbf{a}-\mathbf{g}\|^2+\lambda\mathbf{a}^T\mathbf{F}_i\mathbf{a}$ where λ is the Lagrange multiplier. Setting $\partial J/\partial\mathbf{a}=0$ and solving for the projection $\mathbf{a}_\lambda$, we get

$$\mathbf{a}_\lambda=(\mathbf{I}+\lambda\mathbf{F}_i)^{-1}\mathbf{g}, \quad (19)$$

where λ is to be determined. Since $\mathbf{a}_\lambda$ must satisfy $\mathbf{a}^T\mathbf{F}_i\mathbf{a}=0$, we get

$$\mathbf{g}^T(\mathbf{I}+\lambda\mathbf{F}_i)^{-T}\mathbf{F}(\mathbf{I}+\lambda\mathbf{F}_i)^{-1}\mathbf{g}=0. \quad (20)$$

Now $\mathbf{F}_i$ is real-symmetric of rank four. The eigenvectors of $\mathbf{F}_i$ are orthogonal and there are only four non-zero eigenvalues, say $\gamma_1,\gamma_2,\gamma_3,\gamma_4$. If $\mathbf{E}_i$ and $\boldsymbol{\Gamma}_i$ denote, respectively, the eigenvector and eigenvalues matrices of $\mathbf{F}_i$ then $\mathbf{F}_i=\mathbf{E}_i\boldsymbol{\Gamma}_i\mathbf{E}_i^T$, $\mathbf{E}_i\mathbf{E}_i^T=\mathbf{I}$, and $\mathbf{E}_i^T=\mathbf{E}_i^{-1}$. Using these results in Eq.(20), with $\mathbf{y}\equiv\mathbf{E}_i^T\mathbf{g}$, yields

$$\mathbf{y}^T(\mathbf{I}+\lambda\boldsymbol{\Gamma}_i)^{-1}\boldsymbol{\Gamma}(\mathbf{I}+\lambda\boldsymbol{\Gamma}_i)^{-1}\mathbf{y}=0. \quad (21).$$

Equation (21) can be written in the form $n_i(\lambda)/d_i(\lambda)=0$ where $n_i(\lambda)$ and $d_i(\lambda)$ are numerator and denominator polynomials in the unknown λ.

Since $n_i(\lambda)/d_i(\lambda)=0$ implies $n_i(\lambda)=0$, we finally obtain a polynomial of degree six from which the appropriate value of λ can be recovered. After some tedious but elementary algebra, we obtain

$$\begin{aligned} n_i(\lambda) =\ & y_1^2\gamma_1(1+\lambda\gamma_2)^2(1+\lambda\gamma_3)^2(1+\lambda\gamma_4)^2 \\ &+y_2^2\gamma_2(1+\lambda\gamma_1)^2(1+\lambda\gamma_3)^2(1+\lambda\gamma_4)^2 \\ &+y_3^2\gamma_3(1+\lambda\gamma_1)^2(1+\lambda\gamma_2)^2(1+\lambda\gamma_4)^2 \\ &+y_4^2\gamma_4(1+\lambda\gamma_1)^2(1+\lambda\gamma_2)^2(1+\lambda\gamma_3)^2 \\ &=0. \end{aligned} \quad (22)$$

The roots of $n_i(\lambda)$ can be found numerically, for example the $roots(n)$ function in MATLAB.

We are now in a position to describe the key steps in realizing the projection onto the generic set $C^{(i)}=\{\mathbf{a}\colon \mathbf{a}^T\mathbf{F}_i\mathbf{a}\geq 0\}$. We omit describing such user-determined steps as initializations and tests for convergence.

Step 1. For a given $\mathbf{g}$, check if $\mathbf{g}^T\mathbf{F}_i\mathbf{g}\geq 0$. If $\mathbf{g}^T\mathbf{F}_i\mathbf{g}\geq 0$, $\mathbf{g}$ is already in the set and adjust $i=i+1$. Otherwise, go to step 2.

Step 2. Find the roots of $n_i(\lambda)$ and consider only the real roots. For each real root λ compute $\mathbf{a}_\lambda$ from Eq.(19).

Step 3. For each $\mathbf{a}_\lambda$ as computed in step 2, compute $\|\mathbf{g}-\mathbf{a}_\lambda\|$. The $\mathbf{a}_\lambda$ that gives the *smallest norm* $\|\mathbf{g}-\mathbf{a}_\lambda\|$ is the projection.

Step 4. Go on to the next frequency ω_i by setting $i=i+1$.

Note. For projecting onto a generic set of the form $C^{(i)}=\{\mathbf{a}\colon \mathbf{a}^T\mathbf{F}_i\mathbf{a}\leq 0\}$, the steps are the same except for step 1. Here the appropriate operation is:

Step 1. For a given $\mathbf{g}$, check if $\mathbf{g}^T\mathbf{F}_i\mathbf{g}\leq 0$. If $\mathbf{g}^T\mathbf{F}_i\mathbf{g}\leq 0$, $\mathbf{g}$ is already in the set and adjust $i=i+1$. Otherwise, go to step 2.

5. EXAMPLES AND NUMERICAL RESULTS

In both the following examples, the algorithm used is $\mathbf{a}_{n+1}=\frac{1}{2M}\sum_{i=0}^{M-1}\sum_{k=1}^{2}P_{ki}\mathbf{a}_n$ where M is the number of sets associated with the discrete frequencies see Eq.(18) .

Example 1 - Group Delay Equalization for Chebyshev Filter.

We consider the equalization of the non-linear group delay of a 6th-order Chebyshev II lowpass filter with a 40 dB stopband attenuation and 0.3π stopband cutoff frequency by a 4th-order equalizer. These parameters were chosen to enable us to compare our method with published results. We use the algorithm described above to equalize the phase in the passband $[0,0.2\pi]$. We discretize the range $[0,0.2\pi]$ into 40 samples. We chose the nominal group delay $K=19$ that satisfies Eq.(10) and a tolerance $\delta = 0.5$. Not every value of K in this range will produce a stable filter. We tried few different values of K before the algorithm produced a stable filter for the prescribed tolerance δ. Figure 1 shows the original and the equalized group delay. The VSPM yielded a filter with smaller maximum peak-to-peak fluctuations in group delay than the one designed by eigenfilter method $(\Delta\tau_{VSPM} = 1.0 \text{ versus } \Delta\tau_{EM} = 2.7)$ [3]. The proposed algorithm for this example converged after about 20,000 iteration cycles.

Example 2 - Group Delay Equalization for a Quadratic Phase.

In this example, we chose to equalize the group delay of a filter whose phase is quadratic i.e., $\theta_P(\omega) = -\omega^2$ in the passband $[0, 0.2\pi]$. We discretized the range $[0, 0.2\pi]$ into 40 samples and chose a tolerance $\delta = 0.5$ and a nominal group delay K = 12 that satisfies Eq. (10). Figure 2 shows the group delay before and after compensation. As in example 1, the VSPM yielded a filter with smaller maximum peak-to-peak fluctuation in-group delay than the one designed by the eigenfilter method $(\Delta\tau_{VSPM} = 1.0 \text{ versus } \Delta\tau_{EM} = 8.1)$. Moreover, the eigenfilter method [3] failed to provide any compensation. The VSPM algorithm for this example converged after about 15,000 iteration cycles.

6. REFERENCES

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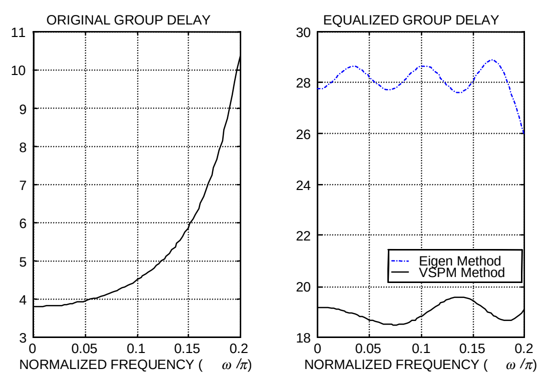

Fig. 1. Original and equalized group delayed of both EM and VSPM designed filters.

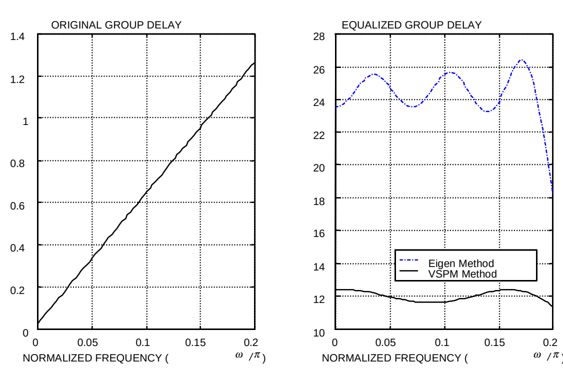

Fig. 2. Original and equalized group delayed of both EM and VSPM designed filters.