NEW RELATIVE MULTIFRACTAL DIMENSION MEASURES

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ABSTRACT

This paper introduces a new class of fractal dimension measures which we call relative multifractal measures. The relative multifractal measures developed are formed through a melding of the Rényi dimension spectrum, which is based on the Rényi generalized entropy, and relative entropy as given with the Kullback-Leibler distance. This new class of multifractal measures is then used to find the relative multifractal complexity differences between two signals, an image and its lossy approximation. It is proposed that relative multifractal measures can be used as the basis for a new form of signal and image quality measure based on signal complexity.

1. INTRODUCTION

Fractal dimension measures arguably date back to the era of Hausdorff and Besicovitch where coverings and box counting methods for measuring objects were proposed. These and succeeding techniques for measuring signal and object complexity, such as Hurst's approach with range-scale statistics [7], led Mandelbrot [12-13] and van Ness [14] to characterize an object through its fractal dimension. The general goal behind a fractal dimension calculation is to form a count N over different scales s in the following power-law relation

$$N \propto s^D. \qquad (1.1)$$

The critical exponent D satisfying this proportionality is, in general, the fractal dimension of the measured object.

In more complex objects and signals, the power-law relation in (1.1) is only capable of characterizing a single level of fractal complexity, the dominant complexity. Extensions of the fractal dimension to multifractal dimensions, i.e. more than one fractal contained in an object, was given by Hentschel and Procaccia [5-6]. The multifractal measures still follow the power-law relation of (1.1), but allow for a moment order on the measurements based on that used in the Rényi generalized entropy H_q [15-17]. This moment order allows the suppression of dominant and enhancement of subdominant fractal components within the more complex multifractal object, revealing specific fractals in the object [8].

A limitation of all of these methods is that the measurement is restricted to a single signal or object, even if the object is multifractal containing multiple fractal components. This paper addresses this limitation by defining a multifractal measure that can compare the *relative* multifractal complexity between two objects. The applicability of this relative multifractal measure is demonstrated with a set of image quality measurement experiments where the relative multifractal complexity difference between an image and its approximations are measured.

2. PROBABILITY MODEL COMPARISON

To perform the desired relative comparisons for this new relative multifractal measure, first consider the probability model comparison given by Landerman *et. al* [10]. As suggested, a system's true probability distribution and its approximation can be compared using the Kullback-Leibler distance [9] as follows

$$0 = D(p^{\text{true}} \| p^{\text{true}}) \le D(p^{\text{true}} \| p^{\text{model}}) \qquad (2.1)$$

where p^{true} is the probability distribution for the true system and p^{model} is the probability set for an approximation to the system. The Kullback-Leibler distance is given by

$$D(u \| v) = \sum_{x \in \chi} u(x) \log \frac{u(x)}{v(x)} \qquad (2.2)$$

for the probability distributions $u(x)$ and $v(x)$.

The use of the Kullback-Leibler distance in (2.1) allows the relative comparison of an approximation to the true model. This concept of model comparison can also be used to determine which of two model approximations are better as follows

$$D(p^{\text{true}} \,\|\, p^{\text{model}(1)}) \le D(p^{\text{true}} \,\|\, p^{\text{model}(2)}) \quad (2.3)$$

For (2.3), model(1) could be considered a better approximation relative to model(2) since the resulting distance to the true model is smaller.

What is now proposed is that the form of relative comparison in (2.1) and (2.3) can be extended from the Kullback-Leibler distance to the Rényi generalized entropy H_q if the following *relative* Rényi entropy is considered

$$RH_q(u \,\|\, v) = \left| \frac{1}{q-1} \log \frac{\sum_{x \in \chi} v(x) \left(\frac{u(x)}{v(x)} \right)^q}{\sum_{x \in \chi} u(x)} \right| \quad (2.4)$$

for the probability distributions $u(x)$ and $v(x)$, and where q is the moment order. It turns out this is nearly the Rényi information, so (2.4) can be expressed as

$$RH_q(u \,\|\, v) = \left| I_q(u \,\|\, v) \right| \quad (2.5)$$

The reason for taking the absolute value in these equations is to maintain the property that the measure is positive such as with the Kullback-Leibler distance where $D(u \,\|\, v) \ge 0$ is always true.

3. RELATIVE MULTIFRACTAL DIMENSION MEASURES

One of the common approaches to measuring multifractal dimensions is to use the Rényi dimension spectrum D_q which is defined as follows [6]

$$D_q = \lim_{s \to 0} \frac{1}{1-q} \frac{\log \sum_{\forall j} p_j^q}{\log(s)} = \lim_{s \to 0} \frac{H_q}{\log(s)} \quad (3.1)$$

where p_j is the probability of the object intersecting with the *volume element* (vel) j, s is the scale of the measurement, and H_q is the Rényi generalized entropy [17].

Culminating from some of our original experiments in multifractal model comparisons [1-3], we have concluded that the comparison of multifractal measures after the calculation of D_q is too prone to calculation errors. In addition as suggested by (3.1), the scale of measurement must be taken to a limit of 0, but this is not possible given a real data set with naturally finite resolution. These two problems makes the comparison of multifractal results after the calculation of (3.1) too sensitive to difficult to control factors. This gives extra credence to the move towards the relative comparison models of (2.1) and (2.3).

Using (2.4) in place of H_q, we can now form a measure in the spirit of the Rényi dimension spectrum from (3.1) as follows

$$RD_q(u \,\|\, v) = \lim_{s \to 0} \frac{\left| \frac{1}{q-1} \log \sum_{x \in \chi_s} v(x) \left(\frac{u(x)}{v(x)} \right)^q \right|}{\log s}. \quad (3.2)$$

This measure will be referred to as the relative Rényi dimension spectrum, RD_q.

Some observations about the behaviour of (3.2) should be noted. The first is that when $q = 0$ the numerator becomes zero, so $RD_{q=0} = 0$. Also, RD_q is a monotonic non-increasing function in q. This is similar to the Rényi dimension spectrum D_q which is also a monotonic non-increasing function in q. These two characteristics of (3.2) give a general idea of the shape of an RD_q versus q curve which has one zero crossing at $q = 0$ and is monotonic non-increasing.

4. EXPERIMENTAL RESULTS

While there are many potential applications and experimental results that could be obtained for a relative multifractal measure, our particular interest in writing this paper is to develop new image quality measures. In particular, we wish to develop image quality measures that consider the overall image feature complexity content at multiple resolutions and are able to characterize the image feature complexity. This is a perfect application of the proposed relative Rényi dimension spectrum in (3.2). To this end, we want to determine how (3.2) behaves for an original image and its lossy reconstruction/approximation.

The experiments performed for this paper consist of seeing how the relative Rényi dimension spectrum RD_q characterizes the relative multifractal complexity differences between original images and their approximations. The experiments use the standard 512×512 8-bit greyscale image of Lena. The original image of Lena acts as the true model and its lossy reconstructions act as the approximated models in terms of (2.1).

For the lossy reconstructions, we decided to use a set of images easy to duplicate by using the discrete wavelet transform (DWT) given by Mallat [11] with the Daubechies 4-tap wavelet filter bank [4]. The specific set of images used are approximated versions of the Lena image through hard thresholding of the wavelet coefficients at a hard threshold level of 2^n for $0 \le n \le 10$, n an

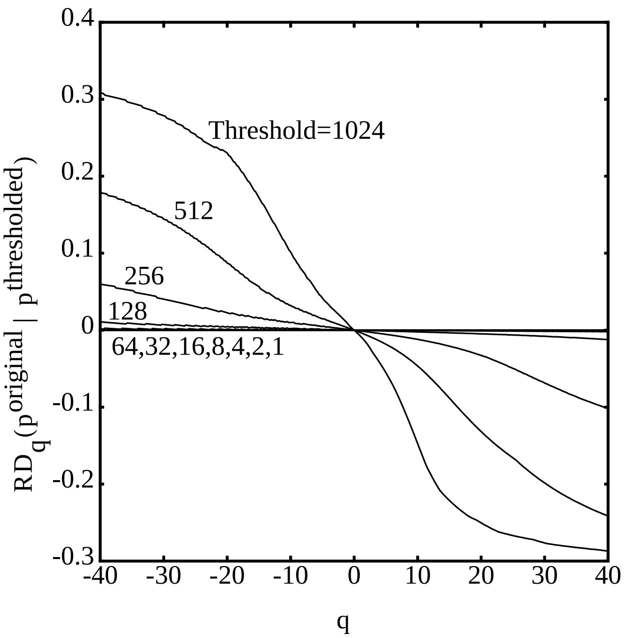

Fig. 1. Plot of relative Rényi dimension spectrum RD_q versus q for Lena approximated from a Daub4 DWT using a hard threshold of 2^n for $0 \le n \le 10$.

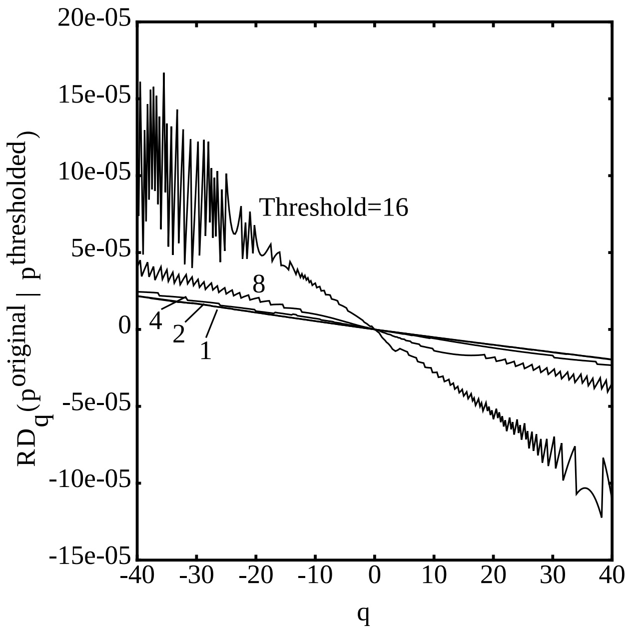

Fig. 2. Plot of relative Rényi dimension spectrum RD_q versus q for Lena approximated from a Daub4 DWT using a hard threshold of 2^n for $0 \le n \le 4$.

integer. It should be understood that the image reconstruction quality for smaller values of n are psychovisually better than for larger values of n.

Using this set of 11 image approximations, the relative Rényi dimension spectrum RD_q is calculated for each approximation, $u(x)$ in (3.2), and the original image, $v(x)$ in (3.2), over $-40 \le q \le 40$. These RD_q versus q results are plotted in Fig. 1 for each of the 11 hard threshold levels.

A number of interesting observations can be made from Fig. 1 that can be generalized to other similar image approximation forms. The first observation is that all of the measured RD_q versus q curves are roughly "centred" around the abscissa origin. Also, the measured RD_q versus q curves appear to follow the monotonic non-increasing predicted behaviour. Another point is that the curves all cross the abscissa origin when $q = 0$, again as predicted. These initial observations help verify that the measurements are consistent with the theoretical behaviour.

The next important consideration from the curves in Fig. 1 is whether they follow the relative model comparisons outlined with (2.1) and (2.3). One note that should be made before addressing this matter is that RH_q and RD_q are two different things. RH_q was designed to be similar to the Kullback-Leibler distance. RD_q is a form of multifractal measure resulting from RH_q, but negative values are possible. Therefore, for RD_q we must use the magnitude when using (2.1) and (2.3). From what is observed in Fig. 1, it is seen that as the image quality improves (i.e. 2^n in the hard thresholding becomes smaller), the RD_q curve converges towards the abscissa origin. Since from (2.1) we know that two identical models will result in $RH_q = 0$ and that RD_q is monotonic non-increasing, it follows that RD_q should converge, as observed, to the abscissa origin as the image approximation improves.

Careful inspection of the plots in Fig. 1 actually shows some unexpected behaviour. This is only visible if we zoom in to see the RD_q versus q curves for the smaller hard thresholds. Figure 2 shows the same RD_q versus q curves for a hard threshold of 1, 2, 4, 8, and 16. The curves actually depart from the theoretical monotonic non-increasing behaviour predicted. The trend to converge to the abscissa origin is still present, but it is clear that some aspect of the calculation causes a non-monotonic behaviour for larger magnitude values of q. In the region around $q = 0$, the curves follow the monotonic behaviour much better. While we have not yet determined the source of this non-monotonic behaviour, it seems to only occur when q has a larger magnitude and when the RD_q values are close to zero. This behaviour may be a result of cumulative errors in the calculation in conjunction with the larger moments of order q. The problem can likely be ignored for the time being since the scale of the RD_q value is many orders of magnitude smaller than at higher threshold values.

5. CONCLUSIONS

We have proposed a new class of multifractal measures which is referred to as relative multifractal measures. In particular, we have developed a new measure based on the Rényi dimension spectrum that allows for the relative comparison of two different probability distributions. These measures have the advantage over other approaches we had developed [1-3] in that the relative comparison of the two probability distributions is done before the main calculation of the multifractal measure. In previous efforts, the comparison was done after application of the multifractal measure which resulted in a measure that is too sensitive to calculation difficulties.

The experiments in this paper have shown that the implemented relative Rényi dimension spectrum largely follows the theoretical behaviour. The experiments performed have shown that there is promising correlation between image quality of a lossy image reconstruction and the convergence of the RD_q to the abscissa origin in a RD_q versus q plot. Some measurement/calculation problems do exist for smaller perceptual differences between an image and its lossy reconstruction, but it is suspected that this is primarily due to larger magnitudes of q in the moment order calculations. Regardless of these issues, there is potential for the relative Rényi dimension spectrum to be used as an image quality measure, or in general as some form of signal quality or signal complexity measure.

ACKNOWLEDGEMENTS

This research was supported in part through a research scholarship and grant from the Natural Sciences and Engineering Research Council (NSERC) of Canada. Also, this work was funded in part from the Telecommunications Research Laboratories (TR*Labs*).

REFERENCES

[1] R. Dansereau and W. Kinsner, “Progressive transmission of images using wavelets: Evaluation using the Rényi generalized entropy,” *Proc. of the Canadian Conf. on Elec. and Comp. Eng. (CCECE’98)*, pp. 273-276, May 1998.

[2] R. Dansereau and W. Kinsner, “Rényi generalized entropy analysis of images from a progressive wavelet image transmission,” *Proc. of the Canadian Conf. on Elec. and Comp. Eng. (CCECE’99)*, 4 pp., May 1999.

[3] R. Dansereau and W. Kinsner, "Progressive image transmission using wavelet packets guided by metrics," special issue of *Mathematical Modelling and Scientific Computing*, vol. 10, 2000. (ISSN 1067-0688)

[4] I. Daubechies, “Orthonormal bases of compactly supported wavelets,” *IEEE Trans. Inform. Theory*, vol. 36, pp. 961-1005, Sept. 1990.

[5] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, “Fractal measures and their singularities: The characterization of strange sets,” *Phys. Rev. A*, vol. 33, no. 2, pp. 1141-1151, Feb. 1986.

[6] H. G. E. Hentschel and I. Procaccia, “The infinite number of generalized dimensions of fractals and strange attractors,” *Physica D: Nonlinear Phenomena*, vol. 8D, no. 1 & 2, pp. 435-444, July 1983.

[7] H. E. Hurst, "Long-term storage capacity of reservoirs," *Trans. of the American Society of Civil Engineers*, vol. 116, pp. 770-799, 1951.

[8] W. Kinsner, “Fractal dimensions: morphological, entropy, spectral and variance classes,” *Technical Report, DEL94-4*, Univ. of Manitoba, 146 pp., May 1994.

[9] S. Kullback and R. A. Leibler, “On information and sufficiency,” *Annals of Math. Stats.*, vol. 22, no. 1, pp. 79-86, 1951.

[10] A. D. Lanterman, J. A. O’Sullivan, and M. I. Miller, “Kullback-Leibler distances for quantifying clutter and models,” *Optical Engineering*, vol. 38, no. 2, pp. 2134-2146, Dec. 1999.

[11] S. G. Mallat, “A theory for multiresolutional signal decomposition: The wavelet transform,” *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 11, pp. 674-693, July 1989.

[12] B. Mandelbrot, *The Fractal Geometry of Nature*, 1982.

[13] B. Mandelbrot, “Self-affine fractals and fractal dimension,” *Physica Scripta*, vol. 32, pp. 257-260, 1985.

[14] B. Mandelbrot and J. van Ness, "Fractional Brownian motions, fractional noises and applications," *SIAM Review*, vol. 10, no. 4, pp. 422-437, 1968.

[15] A. Rényi, “On a new axiomatic theory of probability,” *Acta Mathematica Hungarica*, vol. 6, pp. 285-335, 1955.

[16] A. Rényi, “On the dimension and entropy of probability distributions,” *Acta Mathematica* (Budapest, Hungary), vol. 10, pp. 193-226, 1959.

[17] A. Rényi, “On measures of entropy and information,” *Proc. Fourth Berkeley Symp. Math. Statist. Probab.*, University of California Press, Berkeley, vol. 1, pp. 547-561, 1960.