MIMO FIR EQUALIZERS AND ORDERS

Ravikiran Rajagopal and Lee Potter

Department of Electrical Engineering
The Ohio State University, Columbus, Ohio 43210-1272 USA
ravi{potter}@ee.eng.ohio-state.edu

ABSTRACT

A necessary and sufficient condition is given for the existence of a polynomial left inverse for a polynomial (FIR) system. Additionally, random systems are defined, and a sufficient condition is given for almost sure existence of a polynomial inverse. The two results extend previous work in single-input, multiple-output systems to the case of multiple input systems and to functions of several variables. An algorithm to compute minimal order equalizers is also presented. Corollaries describe calibration of polarimetric, wide-band radar imagery.

1. INTRODUCTION

We consider finite impulse response (FIR) inverse filterbanks for the inversion of FIR distortion filterbanks. Such systems arise naturally in channel equalization for wireless communication systems with mutliple antennas [1], in space object recognition [2] and in polarimetric calibration of radars [3]. These inverse problems have three features in common. First, the systems have multiple outputs. Second, the measurement operator from any one input signal to any one output signal, with the suppression of the other inputs, if any, can be modeled as linear and shift-invariant (LSI). Each of the outputs can be modeled as a superposition of responses from each of the inputs. Third, the distortion on any data point is constrained to be in a finite neighborhood of that point. Thus, the distortion can be modeled as a bank of finite impulse response filters.

2. SYSTEM MODEL

The data and the channel responses may be represented as polynomials in the transform domain $k[z_1, \dots, z_t]$ where t is the dimension of the input data (the FIR convolution takes place in t-space). For our purposes, the z-transform of a data point x may be defined as

$$X(z) = \mathcal{Z}(x) = \sum_{\lambda \in \mathbb{Z}^t_{\geq 0}} x(\lambda) z^\lambda \qquad (1)$$

where $z = (z_1, \dots, z_t)$, $\lambda = (\lambda_1, \dots, \lambda_t)$ is a t-tuple of nonnegative integers, and $z^\lambda = z_1^{\lambda_1} z_2^{\lambda_2} \dots z_t^{\lambda_t}$. Note that the sum above is always finite as all signals we consider have finite extent. Then, the equalization problem can be rephrased as follows.

Let the input polynomials be $X_i(z)$, $1 \leq i \leq m$. We will suppress the independent variable z. Let the output polynomials Y_j be defined as

$$Y_j = \sum_{i=1}^{m} H_{ij} X_i \quad 1 \leq j \leq n \qquad (2)$$

where H_{ij} are polynomials. Now, we seek *polynomials* G_{ij} such that

$$X_i = \sum_{j=1}^{n} G_{ij} Y_j = \sum_{j=1}^{n} \sum_{k=1}^{m} G_{ij} H_{kj} X_k \quad 1 \leq i \leq m$$

In matrix notation, the preceding equation becomes

$$\mathcal{G}\mathcal{H} = I_m \qquad (3)$$

where (notice the nonstandard indexing of the elements of $\mathcal{H}$),

$$\mathcal{H} = \begin{pmatrix} H_{11} & H_{21} & \dots & H_{m1} \\ H_{12} & H_{22} & \dots & H_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{1n} & H_{2n} & \dots & H_{mn} \end{pmatrix} \qquad (4)$$

$$\mathcal{G} = (G_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \qquad (5)$$

and I_m denotes the $m \times m$ identity matrix. Since constants are polynomials too, it is immediate from linear algebra that for solvability of the preceding equation, it is necessary that $n \geq m$.

Prepared through collaborative participation in the Advanced Sensors Consortium sponsored by the U.S. Army Research Laboratory under Cooperative Agreement DAAL01-96-2-0001. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright notation thereon. The views and conclusions contained in this document are those of the authors and should not be interpreted as presenting the official policies, either expressed or implied, of the Army Research Laboratory or the U.S. Government.

3. EXISTENCE

3.1. Necessary and Sufficient Conditions

Theorem 1 *A matrix $\mathcal{H}$ of size $n \times m$, whose elements are polynomials with coefficients in a complete field, has a polynomial left inverse $\mathcal{G}$ iff (1) $n \geq m$ and (2) the ideal of $m \times m$ minors of $\mathcal{H}$ generates the ring of polynomials,* i.e., *there is no zero common to all the $m \times m$ minors of $\mathcal{H}$.*

Proof. Let $\mathcal{H}_i$ denote the $m \times m$ submatrices of $\mathcal{H}$. Let D_i denote the determinant of $\mathcal{H}_i$. If the ideal of $m \times m$ minors of $\mathcal{H}$ generates the ring of polynomials, there exist polynomials a_i such that if $N = \begin{pmatrix} n \\ m \end{pmatrix}$,

$$\sum_{i=1}^{N} a_i D_i = 1 \tag{6}$$

Then, for those $\mathcal{H}_i$ such that $D_i \neq 0$, by Cramer's rule, there exist square polynomial matrices $\mathcal{G}_i$ (adjoints of $\mathcal{H}_i$) such that $\mathcal{G}_i \mathcal{H}_i = D_i I_m$. Thus, there exist $m \times n$ matrices $\tilde{\mathcal{G}}_i$ which are $\mathcal{G}_i$ (with additional zero rows corresponding to those columns in $\mathcal{H}$ which are not in $\mathcal{H}_i$) such that $\tilde{\mathcal{G}}_i \mathcal{H} = D_i I_m$, from which it follows that

$$\sum_{i=1}^{N} (a_i \tilde{\mathcal{G}}_i)\mathcal{H} = \sum_{i=1}^{N} a_i D_i I_m = I_m \tag{7}$$

whence

$$\mathcal{G} = \sum_{i=1}^{N} (a_i \tilde{\mathcal{G}}_i) \tag{8}$$

Conversely, suppose that the $m \times m$ minors of $\mathcal{H}$ do not generate the ring of polynomials. Then, by the weak *Nullstellensatz* [4, p. 169], $\exists w = (w_1, w_2, ...w_t) \in k^t$ such that $D_i(w) = 0, 1 \leq i \leq N$. Then r$ank(\mathcal{H}(w)) < m$. Then, there does not exist $\mathcal{G}(w)$ such that $\mathcal{G}(w)\mathcal{H}(w) = I_m$. Thus, a polynomial left inverse of $\mathcal{H}$ does not exist. ■

For the single-input, multiple-output (SIMO) case in one variable, (*i.e.*, $t = 1$, $m = 1$, $n > 1$), Theorem 1 is the classical Bezout equation [5] for which solution techniques are well known [1, 5, 6]. For the multiple-input, multiple-output (MIMO) case, Theorem 1 reduces the existence of a polynomial inverse to the SIMO case. The constructive proof requires solution of equation (6); therefore, the MIMO problem with mn filters H_{ij} is reduced to a SIMO problem with N filters D_i. For the square case of $m = n$, Theorem 1 states that an FIR equalizer exists iff $\det \mathcal{H}$ is a nonzero constant.

3.2. Computing the Inverse

The constructive proof of Theorem 1 provides means for computing the FIR inverse in the MIMO case. First, a result of Berenstein and Yger [5] provides a tight upper bound on the required filter order for the FIR inverse. Let

$$\psi = \max_{1 \leq i \leq N} \deg(D_i) \tag{9}$$

Proposition 2 *If D_i do not share a common zero, then there exist polynomials a_i of order $2(2\psi)^{t-1}$ such that equation (6) is satisfied ([5], [7, Theorem 5] when $t = 2$).*

Note that, for a given ψ, there exist D_i such that the bounds on the a_i are achieved. However, for a given D_i, there may exist lower order solutions a_i which satisfy equation (6).

Now, after zero padding the D_i so that they all have order ψ, the following linear equations yield the a_i:

$$\begin{bmatrix} \mathbf{D_1} & \mathbf{D_2} & \dots & \mathbf{D_N} \end{bmatrix} \begin{bmatrix} \mathbf{A_1} \\ \mathbf{A_2} \\ \dots \\ \mathbf{A_N} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \tag{10}$$

where $\mathbf{D_i}$ is the convolution matrix determined by D_i, $1 \leq i \leq N$ and $\mathbf{A_i}$ are the vectorized coefficients of a_i. In [8], this same approach is used when $t = 1$, $m = 1$ resulting in Sylvester matrices.

If the equalizer is not restricted to be FIR, then the problem can be solved using techniques analogous to the Smith-McMillan form and the matrix fraction description [6, chapters 2 and 6].

Proposition 3 *If there exists a polynomial left inverse $\mathcal{G}$ whose elements have (possibly) complex coefficients, for an $\mathcal{H}$ whose elements have purely real coefficients, then there exists a left inverse $\hat{\mathcal{G}}$ whose elements have purely real coefficents. Moreover, given $\mathcal{G}$, $\hat{\mathcal{G}} = \frac{(\mathcal{G}+\bar{\mathcal{G}})}{2}$ is one such left inverse, where $\bar{\mathcal{G}}$ denotes the matrix obtained by conjugating the coefficients of each polynomial in $\mathcal{G}$.*

Proof. Polynomials are entire functions. Hence, for H_{kj} on the real manifold $\mathbb{R}^t$, which is a domain of uniqueness,

$$\sum_{j=1}^{n} G_{ij} H_{kj} = \delta_{ik} \quad 1 \leq i, k \leq m \tag{11}$$

$$\sum_{j=1}^{n} \bar{G}_{ij} H_{kj} = \delta_{ik} \Rightarrow \sum_{j=1}^{n} \frac{(G_{ij} + \bar{G}_{ij})}{2} H_{kj} = \delta_{ik} \tag{12}$$

Note that $\frac{(G_{ij}+\bar{G}_{ij})}{2}$ is a holomorphic function as it is a polynomial. The result in equation (12) on $\mathbb{C}^t$ then follows from the uniqueness theorem [9] for holomorphic functions. ■

3.3. Computation of a_i, $t = 1$

The problem of finding $\mathcal{G}$ reduces to finding the polynomials a_i of equation (6). If $t = 1$, there exist standard techniques based on the Euclidean division algorithm to solve for the a_i. Consider the order of the polynomials in the inverse matrix, which depend upon the order of the polynomials a_i.

Lemma 4 *For the case $t = 1$, $m = 1$, and $n = 2$, if there exist a_1, a_2 such that*

$$a_1(z)D_1(z) + a_2(z)D_2(z) = u(z), \tag{13}$$

and if $0 \leq \deg u(z) < \max(\deg D_1(z), \deg D_2(z))$, then $\exists r(z), b(z)$ such that [10]

$$r(z)D_1(z) + b(z)D_2(z) = u(z) \tag{14}$$

where $\deg r(z) < \deg D_2(z)$ and $\deg b(z) < \deg D_1(z)$.

Now, suppose D_1 and D_2 have lengths l_1 and l_2 respectively, and further that they have no common zero. Note that $l_i = \deg D_i + 1, \quad i = 1, 2$. Then, Lemma 4 implies that there exist a_1 and a_2 of lengths atmost $l_2 - 1$ and $l_1 - 1$ respectively such that (13) holds. Then the convolution can be written as

$$\underbrace{\left[\begin{array}{c|c} \underbrace{\mathbf{F_1}}_{(l_1+l_2-2)\times(l_2-1)} & \underbrace{\mathbf{F_2}}_{(l_1+l_2-2)\times(l_1-1)} \end{array} \right]}_{\tilde{\mathbf{F}}} \left[\begin{array}{c} \mathbf{a_1} \\ \hline \mathbf{a_2} \end{array} \right] = \left[\begin{array}{c} 1 \\ \mathbf{0} \end{array} \right]$$

where $\mathbf{F_1}$ and $\mathbf{F_2}$ are the convolution matrices determined uniquely by D_1 and D_2 respectively. Note that the matrix $\tilde{\mathbf{F}}$ is square. Then, since D_1 and D_2 do not share a common zero, the matrix has full rank by Theorem 1, and a_1 and a_2 can be uniquely determined. Thus, there exist unique a_1 and a_2 of minimal order.

Given the polynomials D_i, $1 \leq i \leq n$, the a_i can be calculated recursively. Henceforth, it will be assumed that there is no zero common to all of the D_i and the explicit dependence on z will be dropped. The key elements in the recursion are Lemma 4 and the algorithm for calculating the greatest common divisor (gcd) of two polynomials [4]). The algorithm proceeds as follows:

Let $H_1 = D_1$ and let $H_2 = \gcd(H_1, D_2)$. Then,

$$H_1 = H_2 v_{11} \qquad D_2 = H_2 v_{12}$$

where v_{11} and v_{12} do not share common zeros. Then, from Lemma 4, $\exists u_{11}, u_{12}$ such that $\deg(u_{11}) < deg(v_{12})$, $\deg(u_{12}) < deg(v_{11})$ and

$$u_{11}v_{11} + u_{12}v_{12} = 1 \tag{15}$$

$$\Rightarrow u_{11}H_1 + u_{12}D_2 = H_2 \tag{16}$$

Now, suppose H_k has been defined. Let $H_{k+1} = \gcd(H_k, a_{k+1})$. Then,

$$H_k = H_{k+1}v_{k1} \qquad D_{k+1} = H_{k+1}v_{k2}$$

where v_{k1} and v_{k2} do not share common zeros. Then, from Lemma 4, $\exists u_{k1}, u_{k2}$ such that $\deg(u_{k1}) < deg(v_{k2})$, $\deg(u_{k2}) < deg(v_{k1})$ and

$$u_{k1}v_{k1} + u_{k2}v_{k2} = 1 \tag{17}$$

$$\Rightarrow u_{k1}H_k + u_{k2}D_{k+1} = H_{k+1} \tag{18}$$

If $H_k = 1$ for some k, we set $u_{i1} = 1$ and $u_{i2} = 0$, $i > k$. Now, since $\gcd(H_k, a_{k+1}) = \gcd(a_1, \ldots, a_{k+1})$, $H_N = 1$. Thus we obtain $H_1, \ldots, H_N$, $u_{11}, \ldots, u_{N-1,1}$, and $u_{12}, \ldots, u_{N-1,2}$. Now, going back up the recursion ladder,

$$a_k = u_{k-1,2} \prod_{i=k}^{N-1} u_{i1} \qquad 1 \leq k < N$$

where $u_{0,2} = 1$, $a_N = u_{N-1,2}$ and $a_1 D_1 + \ldots + a_N D_N = 1$.

Since the u_{ij} obtained at each step have minimal order, the filters a_i have minimal orders, which are determined by the ordering of the polynomials D_i. To obtain the absolute minimal filter orders, we minimize over all $N!$ permutations of D_i. The search complexity can be decreased by using a Viterbi decoder-like algorithm. In section IV.B of [11], it was shown that if two filters f_1 and f_2 have the same order π, then there exist inverse filters, each of order $\pi - 1$. The algorithm presented here extends the result with sharper bounds on the filter orders.

4. ALMOST SURE EXISTENCE

Definition 5 *Let $\mathcal{H}$ be as in equation (4). Let ϕ denote a fixed upper bound on the total degree of the polynomials H_{ij}. Suppose that the coefficients of H_{ij} are chosen from a continuous probability density (pdf) Ψ on $\mathbb{C}^{mn\Phi}$, where $\Phi = \sum_{i=0}^{\phi} \begin{pmatrix} i+t-1 \\ t-1 \end{pmatrix}$ Then the system is said to be* ***random****.*

Theorem 6 *Let $\mathcal{H}$ be a matrix of size $n \times m$, whose elements are polynomials H_{ij} in t variables, where $n \geq m$ and $\begin{pmatrix} n \\ m \end{pmatrix} > t$. Suppose each H_{ij} is a linear combination of elements from a set W_{ij} of monomials, and that the coefficients of the linear combinations are chosen randomly from a continuous probability density function Ψ on $\mathbb{C}^{\Omega}$,*

$$\Omega = \sum_{i=1}^{m} \sum_{j=1}^{n} \#W_{ij}$$

where $\#W_{ij}$ denotes the cardinality of W_{ij}. Suppose further that Ψ and W_{ij} are such that atleast one of the minors D_k of size $m \times m$ contains a nonzero constant term almost surely. Then, there almost surely exists a polynomial left inverse $\mathcal{G}$ for $\mathcal{H}$.

Proof. Note that the coefficients of the minors D_k are polynomials of the coefficients of H_{ij}. Without loss of generality, let D_1 have a nonzero constant term. Consider the homogeneous polynomials

$$E_k(z_0, z_1, \ldots, z_t) = z_0^{\deg(D_k)} D_k\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \ldots, \frac{z_t}{z_0}\right)$$

E_k share a common zero in $\mathbb{C}^{t+1} - \{0\}$ whenever D_k share a common zero in $\mathbb{C}^t - \{0\}$. The resolvent $R(\rho_1, \ldots, \rho_{t+1})$ of $t+1$ homogeneous polynomials $\rho_1, \ldots, \rho_{t+1}$ in $t+1$ variables is a polynomial in the coefficients of $\rho_1, \ldots, \rho_{t+1}$ which vanishes iff the polynomials $\rho_1, \ldots, \rho_{t+1}$ share a common zero in $\mathbb{C}^{t+1} - \{0\}$. Then $R(D_1, \ldots, D_{t+1})$ is a polynomial in Ω variables [12, p. 427, Prop. 1.1]. Hence, it vanishes only on an affine variety which is a closed and nowhere dense set of measure zero. Also, the set of coefficients where D_1 vanishes at the origin is a closed linear subspace of $\mathbb{C}^\Omega$ which is a nowhere dense set of measure zero. Thus, for almost every choice of the coefficients of H_{ij}, the minors D_k do not share a common zero. Hence, by Theorem 1, and since the Lebesgue integral on a set of measure zero of an atomless function vanishes, a polynomial left inverse almost surely exists. ■

Corollary 7 *Let $\mathcal{H}$ be as in Definition 5. If $N > t$, and if the region of support of Ψ is not a proper subspace of $\mathbb{C}^\Phi$, the inverse filter bank almost surely exists.*

Corollary 7 has been proved for the cases $t = 1$, $m \geq 1$ [5], and $t = 2$, $m = 1$ [13]. Harikumar and Bresler [7] show that for $t = 2$, $m = 1$, $n > 2$, for almost every choice of distortion filters, there is no common zero shared by these filters. They then present an algorithm for the blind estimation of X_1. The basic idea is that if the filters do not share a common zero, then X_1 must be the greatest common divisor of the Y_j.

5. WIDE-BAND RADAR CALIBRATION

In polarimetric synthetic aperture radar imaging, signals are transmitted and received in two polarizations. By reciprocity, the response of the target is identical in the crosspolarized (HV, VH) channels. The observed responses o_{ab} can be modeled as follows [3]:

$$\begin{bmatrix} o_{hh} & o_{hv} \\ o_{vh} & o_{vv} \end{bmatrix} = \begin{bmatrix} r_{hh} & r_{hv} \\ r_{vh} & r_{vv} \end{bmatrix} \begin{bmatrix} s_h & s_x \\ s_x & s_v \end{bmatrix} \begin{bmatrix} t_{hh} & t_{hv} \\ t_{vh} & t_{vv} \end{bmatrix}$$

where r_{ab} and t_{ab} represent the receive and transmit antenna responses, and where s_a represent the response of the target. Now, assuming that these functions are polynomials, *i.e.*, FIR filters with $t = 2$, we obtain the following system description:

$$\begin{bmatrix} o_{hh} \\ o_{hv} \\ o_{vh} \\ o_{vv} \end{bmatrix} = \begin{bmatrix} r_{hh}t_{hh} & r_{hv}t_{hh} + r_{hh}t_{vh} & r_{hv}t_{vh} \\ r_{hh}t_{hv} & r_{hv}t_{hv} + r_{hh}t_{vv} & r_{hv}t_{vv} \\ r_{vh}t_{hh} & r_{vv}t_{hh} + r_{vh}t_{vh} & r_{vv}t_{vh} \\ r_{vh}t_{hv} & r_{vv}t_{hv} + r_{vh}t_{vv} & r_{vv}t_{vv} \end{bmatrix} \begin{bmatrix} s_h \\ s_x \\ s_v \end{bmatrix}$$

Following the proof of Theorem 6, it can be shown that this system is invertible almost surely if the antenna response functions are drawn from a continuous pdf, as $\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4 > 2$.

6. REFERENCES

[1] E. Moulines, P. Duhamel, J. Cardoso, and S. Mayrargue, "Subspace methods for the blind identification of multichannel FIR filters," in *Proc. ICASSP 1994*, 1994, vol. 4, pp. 573–576.

[2] Xun Du, Stanley C. Ahalt, and Bruce Stribling, "3-D Orientation Vector Estimation for Sub-Components of Space Object Imagery," *Optical Engineering*, vol. 37, no. 3, pp. 798–807, Mar. 1998.

[3] Emre Ertin and Lee C. Potter, "Polarimetric calibration for wide band synthetic aperture radar imaging," *IEE Proceedings - Radar, Sonar and Navigation*, vol. 145, no. 5, pp. 275–280, Oct 1998.

[4] D. A. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*, Springer-Verlag, 1992.

[5] C. A. Berenstein and A. Yger, *Residue Currents and Bezout Identities*, Birkhäuser, Boston, 1993.

[6] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980.

[7] G. Harikumar and Y. Bresler, "Exact image deconvolution from multiple FIR blurs," *IEEE Transactions on Image Processing*, vol. 8, no. 6, pp. 215–8, June 1999.

[8] L. Tong, G. Xu, and T. Kailath, "Blind identification and equalization based on second-order statistics: a time domain approach," *IEEE Transactions on Information Theory*, vol. 40, no. 2, pp. 340–349, March 1994.

[9] Lars Hörmander, *An Introduction to Complex Analysis in Several Variables*, North-Holland, 3rd edition, 1990.

[10] Carlos A. Berenstein and Roger Gay, *Complex variables: An introduction*, Springer Verlag, 1991.

[11] E. Moulines, P. Duhamel, J. Cardoso, and S. Mayrargue, "Subspace methods for blind identification of multichannel FIR filters," *IEEE Transactions on Signal Processing*, vol. 43, no. 2, pp. 516–525, February 1995.

[12] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[13] G. Harikumar and Y. Bresler, "Perfect blind restoration of images blurred by multiple filters: theory and efficient algorithms," *IEEE Transactions on Image Processing*, vol. 8, no. 2, pp. 202–19, February 1999.