ON THE EQUALIZABILITY OF NONLINEAR/TIME-VARYING MULTI-USER CHANNELS

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ABSTRACT

We attack the problem of perfect equalizability of multi-user channels, in which the usual linear time-invariant assumption is dismissed. In the linear, time-invariant case, condition for perfect equalizability is plain and expressed in terms of the column rank of the channel's transfer matrix. Using the module-theoretic approach developped by Fliess, in which the transfer matrix of a time-varying channel as well as the rank of a non-linear channel are clearly defined, we show how the condition obtained in the linear time-invariant case naturally extends to the time-varying and the non-linear cases.

1. INTRODUCTION

With the advent of multi-acces digital communication systems, increasing attention is devoted to the multi-user channels equalization problem, especially in blind case. When the channel is left invertible, (blind) equalization is, conceptually, a well posed problem as it reduces to find the channel left inverse or a stable and causal approximation of it. A number of candidate approaches is now available, including both training sequence, blind and semi-blind versions (see [1] and references therein). The bulk of these approaches relies on a linear time-invariant channel setting in which left invertibility is a generic property as soon as the number of output observations is high enough. Clearly, a multi-input multi-output linear time-invariant channel admits a left inverse whenever the *generic* rank of its transfer matrix equals the number of inputs.

However, several real-time communication channels are time-varying and/or nonlinear [2], [3]. A well known example of a nonlinear channel is given by a digital communication satellite system where nonlinear distorsions are introduced by the down-link amplifiers. Nontheless, little attention has been devoted to the equalization of such channels, espacially in the multichannel case. The main reason for this gap, probably, stems from the fact that the classical concept of "transfer matrix" ceases to make sense in the time-varying case and, even more, in the nonlinear case where the notion of matrix itself is undefined.

Therefore, concerning the equalization of such multichannels, one can not dodge the question of what the term "inverse" refers to.

The module-theoretic approach developped by Fliess [4], [5], [6] gives satisfying answers to this question.

In a linear time-varying situation, there exists several approaches [7], [8], [5], that provide a clear-cut interpretation of the transfer matrix, although the z-transform does not have a straightforward definition. Among these approaches, the module-theoretic one yields an algebraic framework in which time-varying transfer matrices act very naturally.

For the nonlinear context, the module-theoretic approach provides a nice interpretation of the rank of a nonlinear system. This interpretation makes it possible to extend the linear case results to the nonlinear case, in a very natural way. The philosophy of this approach is to transform the nonlinear system into an "equivalent" (in a sense to be clarified later) time-varying but linear system and then to apply the known results of the linear case.

In order to fix the notations, we begin with the linear time-invariant case in section 2. The linear time-varying case is investigated in section 3. The mathematical tools which permit to handle the time-invariant nonlinear setting are presented in section 4, along with a necessary and sufficient condition for a given nonlinear multichannel to be left invertible. Section 5 is devoted to an illustrative example of a single-input two-output digital nonlinear communication system. Finally, we give in section 6 some concluding remarks.

2. LINEAR TIME-INVARIANT EQUALIZABILITY

Let $\mathcal{K}$ be a given field (*e.g.* $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$) and consider the linear discrete-time, time-invariant m-input p-output transmission channel system $\mathcal{S}$ depicted in figure 1 with input $\boldsymbol{u}(n) \in \mathcal{K}^m$. The

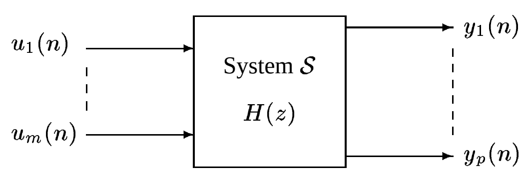

Fig. 1. Input-output Multichannel system

received signal $\boldsymbol{y}(n) \in \mathcal{K}^p$ is modeled as

$$\underbrace{\begin{bmatrix} y_1(n) \\ \vdots \\ y_p(n) \end{bmatrix}}_{\boldsymbol{y}(n)} = \boldsymbol{H}(z) \underbrace{\begin{bmatrix} u_1(n) \\ \vdots \\ u_m(n) \end{bmatrix}}_{\boldsymbol{u}(n)} \quad (1)$$

where we interpret z as the unit delay operator:

$$z\boldsymbol{u}(n) = \boldsymbol{u}(n-1).$$

The channel transfer matrix is assumed stable, causal and finite in duration, say N, so that we may write

$$\boldsymbol{H}(z) = \sum_{k=0}^{N} \boldsymbol{H}_k z^k, \quad \boldsymbol{H}_k \in \mathcal{K}^{p\times m}, \text{ with } \|\boldsymbol{H}_k\| < \inf. \quad (2)$$

In all the sequel, we assume that the number of observations exceeds the number of inputs: $p > m$. We also assume, for technical reasons, that $m(N+1) > p$. Clearly, the multi-user channel $\boldsymbol{H}(z)$ is perfectly equalizable when it admits a stable and causal left inverse. Left invertibility now occurs when $\operatorname{rk}(\boldsymbol{H}(z)) = m$. When the system is time-varying or nonlinear, the situation is however more involved.

We now proceed to translate the left invertibility condition in a module-theoretic framework [4], [6] which will allow us to handle both the time-varying and the non-linear cases in a natural way .

To begin, let $\mathcal{K}[z]$ be the ring of polynomials in the variable z, with coefficients in $\mathcal{K}$ and denote by $[\boldsymbol{y}(n)]$ the $\mathcal{K}[z]$-left module generated by $\boldsymbol{y}(n) = (y_1(n), \dots, y_p(n))$. Each element of this module is of the form $\sum_i f_i(z)y_i(n)$ where each $f_i(z)$ is some polynomial in $\mathcal{K}[z]$. Recall that a family $\{\zeta_1, \dots, \zeta_m\}$ of elements of $[\boldsymbol{y}(n)]$ is said to be $\mathcal{K}[z]$-linearly independent if for any set $\{f_i(z)\}_{i=1}^m$ of m polynomials in $\mathcal{K}[z]$, we have

$$\sum_i f_i(z)\zeta_i = 0 \Longrightarrow f_i(z) = 0 \; \forall i.$$

The rank of $[\boldsymbol{y}(n)]$, subsequently denoted by $\operatorname{rk}[\boldsymbol{y}(n)]$, is defined as the cardinal of any maximum (w.r.t inclusion) $\mathcal{K}[z]$-linearly independent family of $[\boldsymbol{y}(n)]$. Then we have

Proposition 1 ([4], [6]). $\operatorname{rk}[\boldsymbol{y}(n)] = \operatorname{rk}(\boldsymbol{H}(z))$.

In the module-theoretic framework, the left invertibility condition for the channel $\boldsymbol{H}(z)$ thus reads as $\operatorname{rk}[\boldsymbol{y}(n)] = m$. Classically, the computation of $\operatorname{rk}(\boldsymbol{H}(z))$ involves the generalized Sylvester matrix $\mathcal{T}_r(\boldsymbol{H})$ as in

$$\begin{bmatrix} \boldsymbol{y}(n) \\ \vdots \\ \boldsymbol{y}(n-r) \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{H}_0 & \cdots & \boldsymbol{H}_N & & \\ & \ddots & & \ddots & \\ & & \boldsymbol{H}_0 & \cdots & \boldsymbol{H}_N \end{bmatrix}}_{\mathcal{T}_r(\boldsymbol{H})} \begin{bmatrix} \boldsymbol{u}(n) \\ \boldsymbol{u}(n-1) \\ \vdots \end{bmatrix}$$

For r large enough, say $r \geq \frac{m(N+1)-p}{p-m}$, we have

$$\operatorname{rk}(\mathcal{T}_r(\boldsymbol{H})) \leq mr + m(N+1).$$

It is well known that perfect blind equalization is possible when $\mathcal{T}_r(\boldsymbol{H})$ is full colunm rank and moreover, this is achievable using only the second-order statistics of the output [9]. On the other hand, note that one may then always find two integer constants ρ and β such that $\operatorname{rk}(\mathcal{T}_r(\boldsymbol{H})) = \rho r + \beta$, with $\rho \leq m$. The module-theoretic approach provides very meaningful interpretations to these constants, in terms of invertibility. To see this let us introduce the following notations. For $r \geq 0$, define

$$V_r \triangleq \operatorname{span}_{\mathcal{K}}\{y_1(n), \dots, y_p(n), \dots, y_1(n-r), \dots, y_p(n-r)\}$$

where the subscript $\mathcal{K}$ in "$\operatorname{span}_{\mathcal{K}}$" means that the coefficients of the linear combinations are taken in $\mathcal{K}$, and set $V_r \triangleq \{0\}$ for $r < 0$. Thus $(V_r)_{r\in\mathbb{Z}}$ is a non decreasing sequence of $\mathcal{K}$-vector spaces in $[\boldsymbol{y}]$. The sequence is clearly (i) exhaustive: $\cup_{r\in\mathbb{Z}} V_r = [\boldsymbol{y}]$ and (ii) discrete: $V_r = \{0\}$ for r small enough.

Definition 1. *$(V_r)_{r\in\mathbb{Z}}$ is called a filtration of the input-output system with transfer matrix $\boldsymbol{H}(z)$.*

For such a filtration, there exists a degree one $\mathbb{N}[r]$-polynomial (called Hilbert polynomial) of the form $P(r) = \rho r + \beta$ such that for $r \in \mathbb{Z}$ large enough, we have $\dim V_r = P(r)$. Moreover, we have the

Proposition 2 ([10]). $\rho = \operatorname{rk}[\boldsymbol{y}(n)] = \operatorname{rk}(\boldsymbol{H}(z)) = \dim V_{r+1} - \dim V_r$.
The m-input– p-output system with transfer matrix $\boldsymbol{H}(z)$ is therefore left invertible if and only if $\rho = m$.

Remark 1. *Note that for r sufficiently large, the rank of the generalized Sylvester matrix $\mathcal{T}_r(\boldsymbol{H})$ can be expressed as*

$$\operatorname{rk}(\mathcal{T}_r(\boldsymbol{H})) = \rho r + \beta = \rho r + m + \sum_{i=1}^{m} \nu_i$$

where the ν_i's are the Kronecker indices of the rational subspace spanned by the columns of $\boldsymbol{H}(z)$. Therefore, if $\beta = m + \sum_{i=1}^m \nu_i$ is less than $m + mN$, then this would mean that at least one of the conditions: the transfer matrix $\boldsymbol{H}(z)$ (i) is irreducible; (ii) is column-reduced, (iii) has all its column degrees equal to N, is violated (see [9] and references therein).

3. LINEAR TIME-VARYING EQUALIZATION

The system $\mathcal{S}$ is now considered to be time-varying. As in the time-invariant case, the condition for the left invertibility of $\mathcal{S}$ will still be given in terms of its rank. This rank turns out to be defined as the rank of the system transfer matrix. Indeed, though the z-transform does not have a straightforward extention to the non-constant case, there are several approaches [7], [8], [5], that provide a clear-cut interpretation of the transfer matrix for a time-varying system. Among these approaches, the module-theoretic one yields an algebraic framework in which time-varying transfer matrices act very naturally.

Following this approach, the discrete-time, time-varying system $\mathcal{S}$ is defined over a ground field $\mathcal{K}$, equipped with the unit-delay operator z. $\mathcal{K}$ is thus called a difference field. Note that it is no longer a field of constants: the elements of $\mathcal{K}$ do not commute with the operator z. Henceforth, we will use the notation $\mathcal{K}(n)$ to make this dependence explicit.

The system transfer matrix is still represented as in (2), by

$$\boldsymbol{H}(n,z) = \sum_{k=0}^{N} \boldsymbol{H}_k(n) z^k, \quad (3)$$

where the coefficient matrices $\boldsymbol{H}_k(n) \in \mathcal{K}(n)^{p\times m}$ are now dependent of the time index n. As for the time-invariant case, we define $[\boldsymbol{y}(n)]$ to be the $\mathcal{K}(n)[z]$-left module generated by $\boldsymbol{y}(n)$. Proposition 1 then naturally extends to

Proposition 3 ([5]).

$$\operatorname{rk}[\boldsymbol{y}(n)] = \operatorname{rk}(\boldsymbol{H}(n,z)).$$

Note that the definition of $(V_r)_{r\in\mathbb{Z}}$ as the filtration of $\mathcal{K}(n)[z]$-left module $[\boldsymbol{y}(n)]$ associated to the time-varying input-output system $\mathcal{S}$ is still valid and this filtration is such that, for $r \in \mathbb{Z}$ large enough, we have

Proposition 4.

1. $\dim V_r = \rho r + \beta$ *with;*
2. $\rho = \mathrm{rk}[\boldsymbol{y}(n)] = \dim V_{r+1} - \dim V_r$.

Hence

Proposition 5. *The discrete-time, time-varying m-input p-output system $\mathcal{S}$ with output $\boldsymbol{y}(n)$ is left invertible if and only if*

$$\rho = \mathrm{rk}[\boldsymbol{y}(n)] = m.$$

4. NONLINEAR EQUALIZATION

We now investigate the situation where the system $\mathcal{S}$ is nonlinear. This situation is much more involved than its linear counterpart mainly because of the lack of a simple system descriptor, as for example the concept of "transfer matrix". Fortunately, the module-theoretic approach provides a nice intrepretation of the rank of a nonlinear system. This interpretation makes it possible to extend to the nonlinear case, the results of the linear case, in a very natural way. The philosophy of this approach is to transform the nonlinear system into an "equivalent" (in a sense to be clarified below) time-varying but linear system and then to apply the known results of the linear case. This extention requires some elaborated mathematical tools that we introduce hereafter.

Let $\mathcal{K}$ still denotes a ground field *i.e.* $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$. When $\mathcal{K}$ is equipped with the delay operator, it becomes a difference field. Let $\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_l\}$ be a collection of elements on which acts the operator z. Define from $\mathcal{K}$ and $\boldsymbol{\zeta}$ the set of all polynomials

$$P(\zeta_1, \ldots, \zeta_l, z\zeta_1, \ldots, z\zeta_l, z^{\alpha}\zeta_1, \ldots, z^{\alpha}\zeta_l)$$

in $l(\alpha+1)$ variables with coefficients in $\mathcal{K}$ and denote this set as $\mathcal{K}[\boldsymbol{\zeta}]$. One may check that $\mathcal{K}[\boldsymbol{\zeta}]$ is a difference ring with unity, finitely generated by $\boldsymbol{\zeta}$. The corresponding quotient field is a field of difference, subsequently denoted as $\mathcal{K}(\boldsymbol{\zeta})$. Let then $\mathcal{K}(\boldsymbol{y}(n))$ be the field of difference obtained as above, from the system output $\boldsymbol{y}(n)$.

Example 1. *If $\mathcal{K}$ is the field of real numbers, then*

$$\frac{z^2 y_1^3(n) + 7zy_1(n)y_4(n) + 3}{zy_2(n) + 2z^2y_2^2(n)\cdot zy_3^4(n) + 5y_1(n)}$$

is an element of the field of difference $\mathcal{K}(\boldsymbol{y}(n))$.

Definition 2. *A difference field extension L/E is given by two difference fields L and E, equipped with the same difference operator, such that $E \subset L$.*

Definition 3. *An element $\zeta \in L$ is said to be* `transformally algebraic over` *E if and only if, it satisfies an algebraic difference equation with coefficients in E. It means that there exists a polynomial $P(x_0, \ldots, x_l)$ over E such that*

$$P(\zeta, z\zeta, \ldots, z^l\zeta) = 0.$$

Otherwize, this element ζ is said to be `transformally` *E-*`transcendental`.

Definition 4. *Let I be an index set. A subset $\{\zeta_i, i \in I\}$ of L is called transformally E-algebraically independent if, and only if, the set $\{z^j\zeta_i, i \in I, j \in \mathbb{N}\}$ is E-algebraically independent. Such an independent set, which is maximal with respect to inclusion is called a* `transformal transcendence basis` *of the extension L/E. Two such bases have the same cardinality which is called the* `transformal transcendence degree` *of L/E and is denoted by $transf\ tr\deg L/E$.*

In this framework, the rank of a nonlinear system admits a clear-cut definition given by

Definition 5 ([6]). *The rank of the input-output system $\mathcal{S}$ with input $\boldsymbol{u}(n)$ and output $\boldsymbol{y}(n)$, denoted as $\mathrm{rk}\{\mathcal{S}\}$, is defined as*

$$\mathrm{rk}\{\mathcal{S}\} \triangleq transf\ tr\deg \mathcal{K}(\boldsymbol{y}(n))/\mathcal{K}.$$

Now that we have a clear definition for the rank, the left invertibility of the nonlinear system $\mathcal{S}$ mimiks the linear case *i.e.*

Proposition 6. *The input-output system $\mathcal{S}$ with input $\boldsymbol{u}(n)$ and output $\boldsymbol{y}(n)$ is said to be left invertible if, and only if*

$$\mathrm{rk}\{\mathcal{S}\} = m.$$

Our next concern will be how to compute this rank in terms of the rank of a given transfer matrix. Indeed this transfer matrix appears to be that of the tangent system, obtained thanks to the `Kähler differentials` [11]. `Kähler differentials` can be seen as the algebraic version of the usual infinitesimal differential calculus. To proceed, consider the linear application d defined on the difference field $\mathcal{K}(\boldsymbol{y}(n))$ such that:

$$\begin{aligned}
&z(d\zeta) = d(z\zeta) \quad \forall\, \zeta \in \mathcal{K}(\boldsymbol{y}(n)) \\
&d(\alpha\beta) = d(\alpha)\beta + \alpha d(\beta) \quad \forall\, \alpha, \beta \in \mathcal{K}(\boldsymbol{y}(n)) \\
&d(c) = 0 \quad \forall\, c \in \mathcal{K}
\end{aligned}$$

Example 2. *Consider $w(n)$, an element of $\mathcal{K}(\boldsymbol{y}(n))$, where $\mathcal{K} = \mathbb{R}$, given by*

$$w(n) = 2y(n-1)^2 + y(n-1)y(n-2).$$

Then we have

$$dw(n) = [4y(n-1) + y(n-2)]dy(n-1) + y(n-1)dy(n-2).$$

One can readily check that the image of $\mathcal{K}(\boldsymbol{y}(n))$ by d is the $\mathcal{K}(\boldsymbol{y}(n))[z]$-left module generated by $d\boldsymbol{y}(n)$ that we denote by $[d\boldsymbol{y}(n)]$. This module corresponds to a **linear** time-varying system with input $d\boldsymbol{u}(n)$ and output $d\boldsymbol{y}(n)$, representing the tangent system of $\mathcal{S}$. By the equivalence of these two systems, we mean the fundamental result [11]:

$$transf\ tr\deg \mathcal{K}(\boldsymbol{y}(n))/\mathcal{K} = \mathrm{rk}[d\boldsymbol{y}(n)], \tag{4}$$

and we have the

Proposition 7. *The nonlinear input-output system $\mathcal{S}$ with input $\boldsymbol{u}(n)$ and output $\boldsymbol{y}(n)$ is left invertible if, and only if*

$$\mathrm{rk}[d\boldsymbol{y}(n)] = m.$$

5. EXAMPLE

In this example we choose $\mathcal{K} = \mathbb{R}$. Consider the nonlinear 1-input 2-output system $\mathcal{S}$ representing *e.g.* a satellite transmission system [12]. Each path is modelized using a Volterra kernel which results in the input-output relations

$$y_1(n) = h_0 u(n) + h_1 u(n-1) + h_2 u(n)^2 + h_3 u(n-1)^2 \tag{5}$$

$$y_2(n) = cu(n)u(n-1) \tag{6}$$

where the coefficients h_i and c are constant real numbers. The tangent system, obtained using the Kähler differentials is then given by

$$dy_1(n) = \alpha_n du(n) + \beta_n du(n-1) \tag{7}$$

$$dy_2(n) = \gamma_n du(n) + \delta_n du(n-1) \tag{8}$$

where

$$\begin{aligned} \alpha_n &= h_0 + 2h_2 u(n) \\ \beta_n &= h_1 + 2h_3 u(n-1) \\ \gamma_n &= cu(n-1) \\ \delta_n &= cu(n) \end{aligned}$$

Note that this system is linear with time-varying coefficients, and these coefficients belong to the difference field $\mathcal{K}(\boldsymbol{y}(n)[z]$ since they depend on the delayed versions of the input, as for the output $\boldsymbol{y}(n)$.

Recalling the filtration (V_r) associated to the above time-varying system, we have for $r = 0$,

$$V_0 = \mathrm{span}_{\mathcal{K}(\boldsymbol{y}(n))}\{dy_1(n), dy_2(n)\}.$$

Now

$$\dim V_0 = \mathrm{rk}\begin{bmatrix}\alpha_n & \beta_n \\ \gamma_n & \delta_n\end{bmatrix} = 2,$$

so long as $\alpha_n\delta_n - \gamma_n\beta_n \neq 0$. Next, for $r = 1$, we have

$$V_1 = \mathrm{span}_{\mathcal{K}(\boldsymbol{y}(n))}\{dy_1(n), dy_2(n), dy(n-1), dy_2(n-1)\}$$

and $\dim V_1$ is equal to the rank of the Sylvester matrix

$$\begin{bmatrix}\alpha_n & \beta_n & 0 \\ \gamma_n & \delta_n & 0 \\ 0 & \alpha_{n-1} & \beta_{n-1} \\ 0 & \gamma_{n-1} & \delta_{n-1}\end{bmatrix}$$

When $\alpha_n\delta_n - \gamma_n\beta_n \neq 0$ or $\alpha_{n-1}\delta_{n-1} - \gamma_{n-1}\beta_{n-1} \neq 0$, this rank is equal to 3. We may check that if $\alpha_n\delta_n - \gamma_n\beta_n \neq 0$ for all $n \in \mathbb{N}$, then

$$\dim V_r = r + 2, \quad \forall\, r,$$

and therefore, $\dim V_{r+1} - \dim V_r = 1$. We thus deduce that the rank of the system is $\rho = 1$, which means that the nonlinear system $\mathcal{S}$ is left invertible, unless the input u lies on the hypersurface defined by

$$h_0 u(n) - h_1 u(n-1) + 2(h_1 u(n)^2 - h_2 u(n-1)^2) = 0, \forall\, n \in \mathbb{N}.$$

This then shows, as quoted in [3], [13], that not all nonlinear systems admit an inverse and the range of invertibility excludes certain subsets of the input.

The credit of this approach is, among others, that it provides an explicit analytical description of these subsets.

6. CONCLUDING REMARKS

Necessary and sufficient conditions for left invertibility are derived for both linear time-varying and nonlinear multi-input multi-output discrete-time systems. Within the module-theoretic framework, we have shown that both these cases may be seen as natural extensions of the linear time-invariant case.

Left invertibility is a preliminary for the problem of equalization to have a clear mathematical meaning. For a given digital communication system, left invertibility is however not equivalent to perfect equalizability. The fact that a system is left invertible does not guarantee the existence of a perfect equalizer, unless the inverse is stable and causal. The questions pertaining to stability and causality have not been considered in this paper neither those concerning the computation of the inverse.

Finally, note that all the results presented here implicitly assume that the input $\boldsymbol{u(n)}$ is independent. Recall that the input is termed independent if the module $[\boldsymbol{u(n)}]$ is free.

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