CHARACTERIZATION OF SELF-SIMILARITY PROPERTIES OF DISCRETE-TIME LINEAR SCALE-INVARIANT SYSTEMS

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ABSTRACT

Discrete-time linear systems that possess scale-invariance properties even in the presence of continuous dilation were proposed by Zhao and Rao. The principal purpose of this article is to describe results of subsequent investigation which have led to characterization of self-similarity properties of discrete-time signals synthesized by these systems. It is shown that white noise inputs to these linear scale invariant systems, which are unique in DSP literature, produce self similar outputs regardless of the marginal distribution of the noise. In most instances the output is fractional Gaussian. For heavy tailed input distributions, the output is also heavy-tailed and self-similar. It is also shown that it is possible to synthesize statistically self-similar signals whose self-similarity parameters are consistent with those observed in network traffic.

1. INTRODUCTION

The previous work of Zhao and Rao [10]-[14] has shown that it is possible to formulate continuous dilation Linear Scale Invariant (LSI) systems in discrete-time. The basis for their formulation is provided by a definition of scaling or dilation in discrete-time using warping and unwarping functions. Our subsequent work investigating self-similarity properties of signals generated by these systems with white noise inputs has produced important results, the presentation of which is the main purpose of the paper.

A motivation for studying self-similar signals has been provided by the seminal work of Leland *et al* [1] showing that Ethernet traffic is self-similar. Self-similarity has since been found in other types of network traffic including wireless networks [2][3]. Self-similar traffic gives rise to buffering requirements that are different and usually higher from those predicted by Poisson assumptions [4]. Much of the theoretical foundation related to the characterization of statistical self-similarity was laid by Mandelbrot and Van Ness [5] in the context of describing fractional Brownian motion (fBm) and fractional noise. For simulating data such as, for example, network traffic we clearly require synthesis of *discrete-time* self-similar random processes. Several methods have been proposed for generating discrete-time self-similar signals [6],[7],[8],[9]. This paper demonstrates that synthesis of self-similar signals using white noise inputs to our discrete-time LSI produces data whose properties are consistent with that of network traffic.

The paper is organized as follows. Section 2 overviews the discrete-time LSI systems. Simulation results for synthesizing and verification of properties of self-similar data using LSI systems are presented in Section 3 and concluding remarks are made in Section 4.

2. OVERVIEW OF DISCRETE-TIME LSI SYSTEMS

2.1 Time-Scaling

The definition of self-similarity rests on the operation of time scaling or dilation. Whereas it is possible to dilate a continuous-time signal in a continuous fashion, the same cannot be done with discrete-time signals. To avoid this difficulty, Zhao and Rao [10]-[14] define a scaling operator for discrete-time signals that can work with any real-valued scaling factor greater then zero based on a *warping transform* $f(\omega)$ which transforms a discrete-time frequency (ω) to continuous-time frequency (Ω). The inverse transform $f^{-1}(\ )$ defines the continuous-time frequency to discrete-time frequency or unwarping transform. One examples of the warping transform is bilinear transform (BLT)

$$\Omega = f(\omega) \equiv 2\tan(\omega/2)\,. \qquad (1)$$

Using the warping transform defined above and time-frequency scaling property of the continuous time Fourier transform, the scaling operator $S_a[\ ]$ of discrete-time sequence $x(n)$ is defined by

$$y(n) = S_a\left[x(n)\right] = aG^{-1}\left\{X\left[\Lambda_a(\omega)\right]\right\} \qquad (2)$$

where $y(n)$ is the output of the operator, G^{-1} is the discrete-time Fourier transform (DTFT), $\Lambda_a(\omega) = f^{-1}\left[af(\omega)\right]$. The scaling operator is shown in figure 1.

For a stochastic input sequence, if the input $X(n)$ of the discrete-time scaling operator $S_a[\ ]$ is a discrete-time wide-sense stationary random process with power spectral density $P_X(\omega)$, it was shown the output is also wide-sense stationary with power spectral density given by

$$P_Y(\omega) = \frac{a^2 P_X\left[\Lambda_a(\omega)\right]}{\left|\Lambda'_a(\omega)\right|} \qquad (3)$$

where $\Lambda'_a(\omega)$ is the first derivative of $\Lambda_a(\omega)$ with respect to ω.

2.2 Discrete-Time Self-Similarity

Using the discrete-time continuous-dilation scaling operator

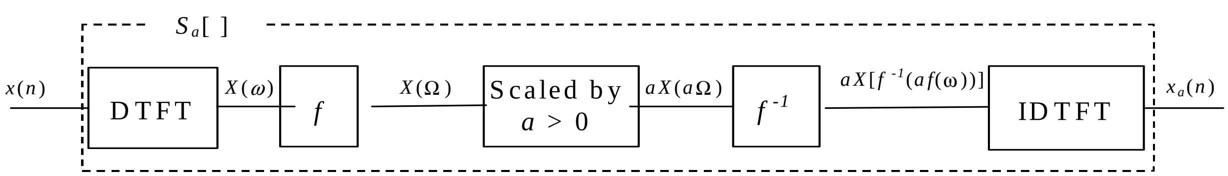

Figure 1. Block diagram of the discrete-time scaling function

S_a[ ] in (2), discrete-time stochastic self-similar signals can be defined as follows: a discrete-time random signal $X(n)$ is said to be self-similar with degree H in the wide-sense if it satisfies the following equations

$$E\left[S_a\left[X(n)\right]\right] = a^{-H} E\left[X(n)\right] \tag{4}$$

and

$$S_{a,a}\left[R_{XX}(n,n')\right] = a^{-2H} R_{XX}(n,n') \tag{5}$$

for any a > 0, where $R_{XX}(n, n')$ is the autocorrelation function of the sequence $X(n)$. For a discrete-time wide-sense stationary random process, the condition of self-similarity simply reduces to

$$\frac{P_X\left[\Lambda_a(\omega)\right]}{\left[\Lambda'_a(\omega)\right]} = a^{-2H-2} P_X(\omega) \tag{6}$$

where $P_X(\omega)$ is the power spectral density of the signal. Therefore, a stationary random process $X(n)$ whose power spectral density satisfies (6) is a self-similar signal in the statistical sense. Zhao and Rao suggested the next power spectrum for the density.

$$P_X(\omega) = \frac{\left|f(\omega)\right|^r}{\left|f'(\omega)\right|} \tag{7}$$

where $f'(\omega)$ is the first derivative of f with respect to ω.

From (6), (7) and $\Lambda_a(\omega) = f^{-1}\left[af(\omega)\right]$,

$$\frac{P_X\left[\Lambda_a(\omega)\right]}{\left[\Lambda'_a(\omega)\right]} = a^{r-1} P_X(\omega). \tag{8}$$

Thus, $X(n)$ is a self-similar random process with $H = -(r + 1)/2$.

If the power spectral density $Px(\omega)$ satisfies the *Paley-Wiener condition*, the density can be factorized as a product $L(\omega)L^*(\omega)$ and by passing white noise through a linear system with frequency response L(ω), the corresponding stochastic self-similar process can be generated.

The power spectral density for the BLT is

$$P_X(\omega) = \frac{\left|f(\omega)\right|^r}{\left|f'(\omega)\right|} = 2^r\left[\frac{1-\cos^2(\omega/2)}{\cos^2(\omega/2)}\right]^{r/2}\cos^2(\omega/2) \tag{9}$$

and this was known to satisfy the *Paley-Wiener condition*.

Let $z = e^{j\omega}$, then $P_X(\omega)$ transforms to

$$P_X(z) = L(z)L(z^{-1}) \tag{10}$$

where the causal part $L(z)$ is

$$L(z) = 2^{r/2-1}(1-z^{-1})^{r/2}(1+z^{-1})^{1-r/2} \tag{11}$$

Note that the spectrum is rational only for integer value of r.

The corresponding impulse response of is a causal filter whose coefficients are given by

$$l_1(n) = \begin{cases} 1 & n = 0 \\ (-1)^n (r/2)\sum_{k=0}^{n}\frac{(r/2-k+1)_{n-1}}{k!(n-k)!} & n > 0 \end{cases} \tag{12}$$

where $(\ )_n$ is the *Pochammer's symbol* defined as

$$(u)_0 \equiv 1 \tag{13}$$

and

$$(u)_v \equiv u(u+1)(u+2)\cdots(u+v-1) = \frac{\Gamma(u+v)}{\Gamma(u)} \tag{14}$$

The impulse response corresponding to $L_2(z)$ is a 2-tap filter with coefficients given by

$$l_2(0) = l_2(1) = 2^{r/2-1} \tag{15}$$

The overall impulse response $l(n)$ corresponding to the system transfer function given in (11) can be represented by two cascaded filters $l_1(n)$ and $l_2(n)$.

2.3 LSI System

A linear scale-invariant (LSI) system is a linear operator L{ } whose output is invariant to scale changes of the input signals, that is,

$$y(n) = L\{x(n)\} \Rightarrow S_a\left[y(n)\right] = L\{S_a\left[x(n)\right]\} \tag{16}$$

where $x(n)$ and $y(n)$ are the input and output sequence respectively.

A discrete-time causal LSI system for a given $x(n)$ can be defined similar to the continuous-time case [15]. Let $h(k)$ be any one-dimensional discrete-time sequence. The discrete-time causal LSI system is defined by the following relationship :

$$y(n) = \sum_{k=1}^{\infty} h(k) S_k\left[x(n)\right]/k \tag{17}$$

The output of the system is the sum of a series of dilation of the input sequence by k that are linearly weighted by $h(k)/k$.

If the input of the LSI system is a discrete-time stochastic self-similar signal with degree H, then the output is also a stochastic, self-similar signal with degree H [10]-[12]. In addition, if the input to a discrete-time LSI system is a discrete-time wide-sense stationary random process, the output of the system is non-stationary due to the fact that the system is time-varying. Using this property, a non-stationary self-similar random signal with parameter $H = -(r + 1) / 2$ can be generated by first generating a discrete-time self-similar random process with degree H by passing zero-mean white noise through a linear system with a frequency response given by (11), and then passing the signal thus obtained through a discrete-time LSI system. Note that the choice of the one dimensional function $h(k)$ in the discrete-time LSI system is arbitrary. This provides flexibility in signal construction. $h(k)$ can be chosen so that the output of the system has certain properties as desired.

3. EXPERIMENTAL CHARA-CTERIZATION OF SELF-SIMILARITY

Given data that are nominally self-similar, the degree of self-similarity H can be estimated in several different ways [1][18]. Three methods are used here. The first method, *the aggregated variance method* relies on the slowly decaying variance of a self-similar series.

$$\text{var}(X^{(m)}) \sim a_2 m^{-\beta}, \text{ as } m \to \infty, \text{ with } 0 < \beta < 1 \tag{18}$$

where, $X^{(m)}$ denote a new time series by averaging $X(n)$ over m non-overlapping sub-blocks and $\beta = -2H$. The degree of self-similarity H can be obtained by drawing corresponding log-log plot and estimating the slope. The second method, *the R/S plot*, uses the fact that for a self–similar dataset, the rescaled range or *R/S* statistic grows according to a power with exponent H as a function of the number of points included *(n)*. If the process is self-similar, *R*/*S* statistic has the following property.

$$E\left[R(n)/S(n)\right] \sim an^{1+H} \tag{19}$$

Thus the *R*/*S plot* on a log-log plot has slope that is an estimate of H. The third approach, the *periodogram* method, uses the slope of the power spectrum of the series as frequency approaches zero. The slope of the log–log plot of the *periodogram* is -1-$2H$.

In order to synthesize the discrete-time self-similar signal, we applied several types of white noise to the filter in (11) with r = –0.6 (H = -0.2). Figure 2 (a), (b) and (c) show the variance-time plot, the pox plot of *R/S*, and the *periodogram* plot that confirm to the observed self-similar properties of Ethernet traffic in figure (7). The synthesized self-similar signals from various inputs such as white Gaussian, uniform, and Pareto distribution are depicted in figure 3. For the heavy-tailed case we chose the simple heavy-tailed Pareto distributed signal, with probability density function

$$p(x) = \alpha k^{\alpha} x^{-\alpha-1}, \ \alpha, \ k > 0, \ x \geq k. \tag{20}$$

The cumulative distribution of (20) is given by

$$F(x) = P[X \leq x] = 1 - (k/x)^{\alpha}. \tag{21}$$

The results, shown in Figure 3, suggest that the output self-similar signal has Gaussian characteristics regardless of the input of the system. In addition, the signal generated from the Pareto distribution shows heavy tailed characteristics. Figure 4(a) plots the autocorrelation function of fractional Gaussian given by

$$\gamma(h) = 2^{-1}\left\{(h+1)^{2H'} - 2h^{2H'} + |h-1|^{2H'}\right\}, \ h \geq 0 \tag{22}$$

The plot of the autocorrelation of the output of our system for white noise input is shown in Figure 4(c). The autocorrelation function decays hyperbolically which confirms its agreement with the decaying characteristic of the fractional Gaussian noise. Figure 5 gives the relationship between the fractional Gaussian noise (FGN) and the generated self-similar signal for different r-values and H'=-(r+1)/2. We get a linear plot with a slope close to 1 again confirming that the output is fractional Gaussian. The slope of the line depends on the *H'* value and the relationship between the system Hurst parameter and the conventional parameter (H) is *H'*=*H*+1. Figure 6 shows the original plots of [1] obtained from Ethernet traffic data. Their closeness to the plots in Figure 2 confirm that it is possible to synthesize data with white noise driven LSI models that conform to network traffic characteristics.

4. CONCLUSION

The discrete-time LSI systems proposed previously by Zhao and Rao provide a potential tool for the analysis and simulation of natural self-similar signals because of their scale invariant property (even though they are time-varying in general) in continuous scale and flexibility in the choice of the 1-D kernel. The paper has provided an empirical demonstration of the fact that white noise driven discrete-time LSI systems can be used to synthesize self-similar sequences with specified value of the *H* parameter. The outputs of these systems are fractional Gaussian for different types of white noise inputs. The explanation for the non-dependence of the output distribution on the input distribution is provided by the central limit theorem. However, the outputs exhibit a heavy-tailed distribution for heavy-tailed inputs. The systems are capable of synthesizing data consistent with the self-similarity that has been documented in network traffic. The discrete-time LSI systems are multi-parametric and are influenced by more than the Hurst exponent. We believe the discrete-time LSI system formulation occupies a place in the study of scale-invariance and self-similarity that corresponds to the position of linear discrete-time time-invariant systems in the study of stationary random processes. A challenging area for further research will be to investigate physical interpretation of LSI models and transformation tools for such systems analogous to Fourier analysis.

ACKNOWLEDGMENT

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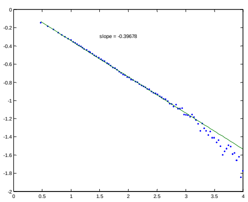
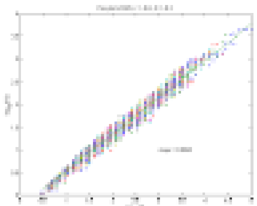
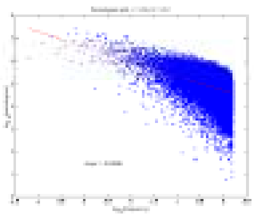


(a) varaince plot

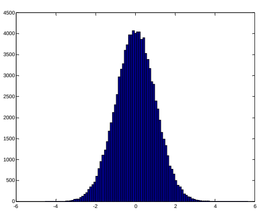


(b) pox plot

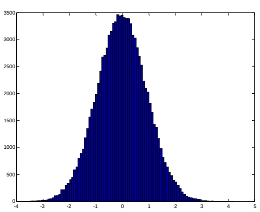


(c) periodogram plot

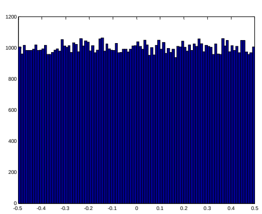
Figure 2. Statistical properties of synthesized self-similar signal (system input: white Gaussian noise, *r* = -0.6, *H* = -0.2)



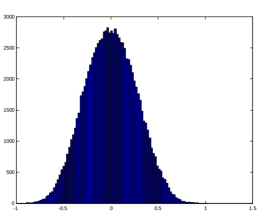
(a) input: Gaussian



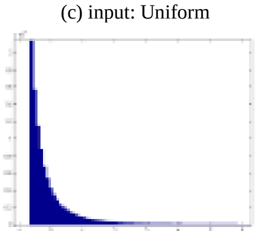
(b) output



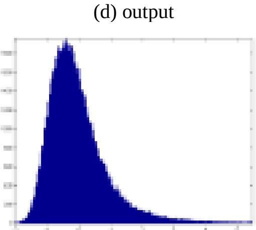
(c) input: Uniform



(d) output

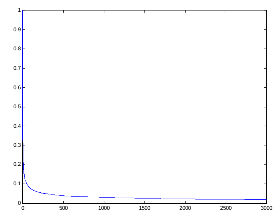


(e) input: Pareto

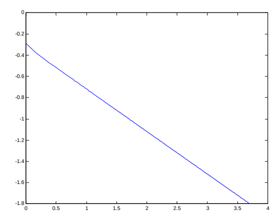


(f) output

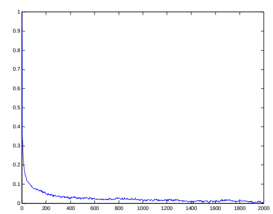
Figure 3. Histograms of self-similar signals from several types of white noise, (a),(b) input: Gaussian, (c), (d) uniform, (e),(f) Pareto distribution



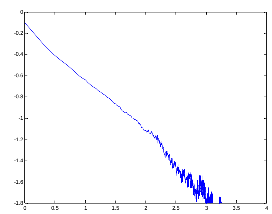
(a)



(b)



(c)



(d)

Figure 4. (a) autocorrelation of the fractional Gaussian noise, (b) log-log scale of (a), (c) autocorrelation of the self-similar signal, (d) log-log scale of (c)

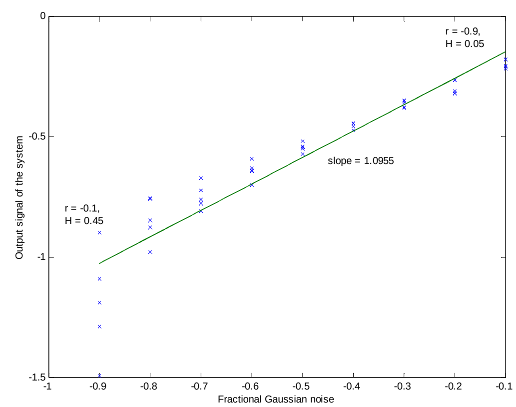
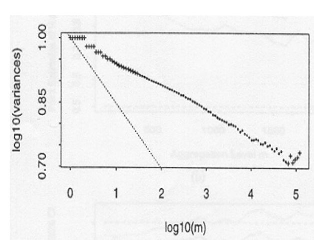
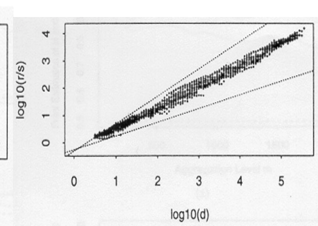


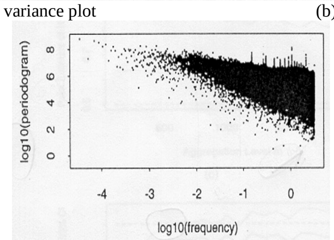
Figure 5. Estimated vs. true slopes of log-log scaled autocorrelation functions (*r* = -0.1 ~ -0.9, *H* = -0.45~-0.05)



(a) variance plot



(b) pox plot



(c) periodogram plot

Figure 6. Statistical property of practical Ethernet network traffic (*H* ≈ -0.2, *r* ≈ -0.6). After Leland et al.